

# Characters of Fredholm modules and a problem of Connes

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A large part of research in noncommutative geometry is devoted to various far reaching generalizations of the Atiyah-Singer index theory of elliptic operators on compact manifolds. Following Kasparov [Ka], the notion of an elliptic operator on a manifold is replaced by that of a Fredholm module (or  $K$ -cycle) over an algebra of operators on Hilbert space. A number of index theorems for Fredholm modules associated to geometric data have been obtained by Kasparov [Ka], Connes (see [Co2] and the literature cited therein) and others. In order to handle concrete index problems it is not only necessary to dispose of an index theorem but also to provide an index formula which allows the explicit calculation of indices. The classical index formula of Atiyah and Singer is obtained from the index theorem by applying the Chern character in topological  $K$ -homology. This motivates the search for character formulas of Fredholm modules ( $K$ -cycles) that define a Chern character on the  $K$ -homology groups of an algebra of operators.

It was this search for a Chern character in  $K$ -homology which led A. Connes to the invention of cyclic cohomology [Co1]. He obtained various explicit character formulas [Co1], [Co], [Co2] which depend on the degree of analytic regularity (summability) of the given Fredholm module. Whereas the classical index problems of Atiyah-Singer are all finitely summable in the sense of [Co1] there are many examples of Fredholm modules over (noncommutative) algebras which are infinite-dimensional (not finitely summable). Characters of Fredholm modules have been calculated in many finitely summable cases, but as far as the author knows the character of a  $\theta$ -summable, infinite dimensional (unbounded) Fredholm module [Co] has not yet been determined in a single case.

Typical examples of infinite dimensional (unbounded) Fredholm modules are modules over dense subalgebras of the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  of a discrete nonamenable group  $\Gamma$  [Co5], [Co2]. A particularly interesting example is presented by Connes in [Co3], [Co2], where he constructs an infinite dimensional unbounded Fredholm module  $\mathcal{E}_\Gamma$  over the group ring of a discrete subgroup  $\Gamma$  of a real semisimple Lie group  $G$ . This module is closely related to Kasparov's  $\gamma$ -element [Ka1]. In [Co3], [Co2] Connes makes several steps towards the calculation of the character of  $\mathcal{E}_\Gamma$  and predicts that it should be cohomologous to the canonical trace on  $C_r^*(\Gamma)$ . He poses the verification as a problem [Co3], p.83 and notes that a positive solution would imply the Kadison-Kaplansky idempotent conjecture for  $\Gamma$  [Co1], [Co2].

In this paper we solve Connes' problem for cocompact discrete subgroups  $\Gamma$  of semisimple Lie groups  $G$  of real rank one. (This restriction is necessary because we have to make use of Jolissaint's rapid decay property for  $\Gamma$  [CM].) To be more precise we prove the

**Theorem 0.1.** *Let  $\Gamma$  be a discrete cocompact subgroup of the semisimple Lie group  $G$  of real rank one. Let  $\mathcal{E}_\Gamma$  be Connes' unbounded  $\theta$ -summable Fredholm module over  $\mathcal{A}(\Gamma) \subset C_r^*(\Gamma)$  [Co2] which represents the  $\gamma$ -element  $\gamma_\Gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$  [Ka1]. Then the entire cyclic character cocycle [Co]*

$$Ch_\epsilon(\mathcal{E}_\Gamma) \in CC_\epsilon^0(\mathcal{A}(\Gamma))$$

*of  $\mathcal{E}_\Gamma$  is cohomologous in local cyclic cohomology [Pu2] to the canonical trace on  $\mathcal{A}(\Gamma)$*

$$[Ch_\epsilon(\mathcal{E}_\Gamma)] = [\tau_{can}] \in HC_{loc}^0(\mathcal{A}(\Gamma))$$

There are two main ingredients in the proof of the theorem. The first is a comparison theorem for character formulas attached to a given Fredholm module. The second is the partial calculation of the equivariant bivariant Chern-Connes character of the  $\gamma$ -element obtained in [Pu3].

The largest part of this paper is concerned with a detailed analysis of the relation between Connes' explicit character formula for an unbounded  $\theta$ -summable Fredholm module over an algebra of operators [Co] and the bivariant Chern-Connes character for bounded Kasparov-bimodules over  $C^*$ -algebras [Pu1]. The following comparison theorem is the second main result of the paper.

**Theorem 0.2.** *Let  $\mathcal{E}$  be an unbounded  $\theta$ -summable Fredholm module over the algebra  $A$ . Let  $\overline{A}$  be the enveloping  $C^*$ -algebra of  $A$  and let  $[\mathcal{E}] \in KK(\overline{A}, \mathbb{C})$  be the  $K$ -homology class corresponding to the given module (it may be represented by any bounded Fredholm module attached to  $\mathcal{E}$  by functional calculus). Denote by  $Ch_\epsilon$  Connes' character of unbounded  $\theta$ -summable Fredholm modules in entire cyclic cohomology and let  $ch_{biv}$  be the bivariant Chern-Connes character on  $KK$ -theory with values in (bivariant) local cyclic cohomology. Then the images of the characters in the diagram*

$$\begin{array}{ccccc} HC_\epsilon^0(A) & \rightarrow & HC_{loc}^0(A) & \leftarrow & HC_{loc}^0(\overline{A}) \\ [Ch_\epsilon(\mathcal{E})] & \rightarrow & * & \leftarrow & ch_{biv}([\mathcal{E}]) \end{array}$$

*coincide.*

We make a few comments about this result. The advantage of Connes' character in entire cyclic cohomology lies in the fact that it provides a completely explicit character formula. However it turns out to be very rigid in several ways. An unbounded Fredholm module over  $A$  defines a  $K$ -homology class over the enveloping  $C^*$ -algebra  $\overline{A}$ . The entire cyclic cohomology groups of  $A$  and  $\overline{A}$  will usually differ from each other. So if two  $K$ -cycles define the same  $K$ -homology class but have

different domains it may be impossible to compare their characters because they lie in completely different cohomology groups. Even if two homotopic  $K$ -cycles have the same domain  $A \subset \overline{A}$ , their characters will not necessarily be cohomologous. An example is given by the pull-back along two different evaluation maps  $C[0, 1] \longrightarrow \mathbb{C}$  of an unbounded Fredholm module with nonzero index over  $\mathbb{C}$ . This happens because the entire cyclic theory is not a (continuous) homotopy functor, as was shown by Khalkhali [Kh].

The bivariant Chern-Connes character is of a very different type. Its existence is established by an abstract category theoretic argument based on the axiomatic characterization of bivariant  $K$ -theory [Hi]. Because the definition of the Chern-Connes character involves excision in local cyclic cohomology, no explicit character formulas exist. This is the main drawback of this character. On the other hand the Chern-Connes character of a  $K$ -cycle depends by construction only on its  $K$ -homology class. Moreover this character has excellent functorial properties due to its multiplicativity with respect to the Kasparov-product on bivariant  $K$ -theory.

Thus the theorem above provides a link between two characters of very complementary nature. This explains the interest in such a result.

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## 1 Characters of Fredholm modules

We collect some well known material about Fredholm modules, cyclic cohomology and Chern characters of Fredholm modules. The only new result is a description of Connes' Chern character of an unbounded  $\theta$ -summable fredholm module [Co] in terms of the Cuntz-Quillen picture of cyclic cohomology. This is slightly different

from Connes' point of view and emphasizes the similarities of Connes' character and the bivariant Chern-Connes character of Cuntz [Cu1], [Pu1], [Pu2].

## Fredholm modules

We recall for the convenience of the reader the definition of (un)bounded Fredholm modules over an algebra.

**Definition 1.1.** [Ka],[Co1]. A bounded even Fredholm module over a  $C^*$ -algebra  $A$  is a quadruple  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, F)$  consisting of

- a separable Hilbert space  $\mathcal{H}$ .
- a selfadjoint operator  $\epsilon \in \mathcal{L}(\mathcal{H})$  satisfying  $\epsilon^2 = 1$ . It defines a  $\mathbb{Z}/2\mathbb{Z}$ -grading  $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$  of  $\mathcal{H}$  which is given by the  $\pm 1$ -eigenspaces of  $\epsilon$ .
- a representation  $\rho : A \longrightarrow \mathcal{L}(\mathcal{H})$  which is even in the sense that it commutes with the grading operator  $\epsilon$ .
- an operator  $F \in \mathcal{L}(\mathcal{H})$  which satisfies

$$\begin{aligned} \rho(a)(F^2 - 1) &\in \mathcal{K}(\mathcal{H}), & \rho(a)(F - F^*) &\in \mathcal{K}(\mathcal{H}), \\ \rho(a)(\epsilon F + F\epsilon) &\in \mathcal{K}(\mathcal{H}), & [F, \rho(a)] &\in \mathcal{K}(\mathcal{H}) \end{aligned} \tag{1.1}$$

for all  $a \in A$ , where  $\mathcal{K}(\mathcal{H})$  denotes the ideal of compact operators in  $\mathcal{L}(\mathcal{H})$ . An odd Fredholm module is defined similarly by forgetting the grading of  $\mathcal{H}$ .

The set of homotopy classes of even (odd) Fredholm modules over a  $C^*$ -algebra  $A$  carries a natural abelian group structure [Ka]. These are the  $K$ -homology groups  $K^*(A) = KK_*(A, \mathbb{C})$  of  $A$  [Ka].

**Definition 1.2.** [Ka],[Co1]. An unbounded even Fredholm module over an algebra  $A$  is a quadruple  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  consisting of

- a separable Hilbert space  $\mathcal{H}$ .
- a selfadjoint operator  $\epsilon \in \mathcal{L}(\mathcal{H})$  satisfying  $\epsilon^2 = 1$ . It defines a  $\mathbb{Z}/2\mathbb{Z}$ -grading of  $\mathcal{H}$  given by the  $\pm 1$ -eigenspaces of  $\epsilon$ .
- a representation  $\rho : A \longrightarrow \mathcal{L}(\mathcal{H})$  which is even in the sense that it commutes with the grading operator.
- an unbounded selfadjoint operator  $\mathcal{D}$  on  $\mathcal{H}$  which satisfies

$$\epsilon \mathcal{D} + \mathcal{D} \epsilon = 0, \quad \rho(a)(1 + \mathcal{D}^2)^{-1} \in \mathcal{K}(\mathcal{H}), \quad [\mathcal{D}, \rho(a)] \in \mathcal{L}(\mathcal{H}) \tag{1.2}$$

for all  $a \in A$ . An odd Fredholm module is defined similarly by forgetting the grading of  $\mathcal{H}$ .

If  $\mathcal{E}$  is an unbounded Fredholm module with domain  $A$ , then we suppose that  $A$  is complete with respect to the norm

$$\|a\|_A = \|\rho(a)\|_{\mathcal{L}(\mathcal{H})} + \|\mathcal{D}, \rho(a)\|_{\mathcal{L}(\mathcal{H})} \quad (1.3)$$

Therefore  $A$  is a Banach algebra.

Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an (even) unbounded Fredholm module over  $A$ . Denote by  $\overline{A}$  the enveloping  $C^*$ -algebra of  $\rho(A) \subset \mathcal{L}(\mathcal{H})$  and let  $f$  be a continuous function on the real line such that  $\lim_{t \rightarrow +\infty} f(t) = +1$ ,  $\lim_{t \rightarrow -\infty} f(t) = -1$ . Then the quadruple  $\mathcal{E}' = (\mathcal{H}, \rho, \epsilon, f(\mathcal{D}))$  defines a bounded (even) Fredholm module over  $\overline{A}$ . Its  $K$ -homology class depends only on  $\mathcal{E}$  and not on the choice of  $f$ . It is called the  $K$ -homology class associated to the given unbounded Fredholm module.

It is possible to impose certain normalization conditions on bounded Fredholm modules without changing their  $K$ -homology class. So one can achieve by a suitable homotopy that

$$F^2 - 1 = F - F^* = \epsilon F + F\epsilon = 0$$

We will however demand less and will suppose henceforth that the considered bounded Fredholm modules satisfy apart from (1.1) the normalization condition

$$F^2 = 1, \quad F - F^* \in \mathcal{K}(\mathcal{H}), \quad \epsilon F + F\epsilon \in \mathcal{K}(\mathcal{H}) \quad (1.4)$$

If  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  is an unbounded even Fredholm module then

$$\mathcal{E}(z) = (\mathcal{H}, \rho, \epsilon, F(z)), \quad F(z) = \frac{\mathcal{D} + z^{\frac{1}{2}}\epsilon}{(\mathcal{D}^2 + z)^{\frac{1}{2}}}, \quad z \in \mathbb{C} \setminus \mathbb{R}_- \quad (1.5)$$

is a holomorphic family of bounded Fredholm modules satisfying (1.4) and representing the  $K$ -homology class of  $\mathcal{E}$ . Here  $z \rightarrow z^{\frac{1}{2}}$  denotes the branch of the square-root function on the domain  $\mathbb{C} \setminus \mathbb{R}_-$  which takes the value  $+1$  at  $1$ .

The character formulas for Fredholm modules which we are going to study often require the involved operators to satisfy certain regularity conditions. The notion of finite summability, i.e.  $\text{Trace}((1 + \mathcal{D}^2)^{-p}) < \infty$  for  $p \gg 0$  [Co1] is well known but rather restrictive. For example there exists no finitely summable unbounded Fredholm module over a dense subalgebra of the reduced group  $C^*$ -algebra of a nonamenable discrete group [Co5]. Most known examples of Fredholm modules satisfy however the following much weaker regularity condition.

**Definition 1.3.** [Co] An unbounded Fredholm module  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  is  $\theta$ -summable if

$$\text{Trace}(e^{-t\mathcal{D}^2}) < \infty, \quad \forall t > 0 \quad (1.6)$$

**Remark 1.4.** *A. Connes works in [Co2] also with Fredholm modules which satisfy  $\text{Trace}(e^{-t\mathcal{D}^2}) < \infty$  only for  $t \gg 0$ . His character formula makes perfectly sense for such modules. The methods of this paper apply however only under the condition that the whole heat semigroup of  $\mathcal{D}$  is of trace class.*

A basic invariant of an even Fredholm module is its index

$$Index(\mathcal{E}) = \dim Ker(\mathcal{D}|_{\mathcal{H}_+}) - \dim Ker(\mathcal{D}|_{\mathcal{H}_-}) \quad (1.7)$$

The index of a Fredholm module is homotopy invariant and depends therefore only on its  $K$ -homology class.

## Cyclic cohomology theories

We recall some well known facts about cyclic complexes taken from Connes [Co],[Co1],[Co2], Cuntz and Quillen [CQ],[CQ1],[CQ2] and [Pu1],[Pu2]. We only treat cohomology because the corresponding homology groups will play no role in this paper.

Let  $A$  be a complete, locally convex algebra over the complex numbers with jointly continuous multiplication. The universal complete, locally convex, differential graded algebra  $\Omega A$  over  $A$  is given by

$$\begin{aligned} \Omega A &= \bigoplus_{n=0}^{\infty} \Omega^n A \simeq A \oplus \bigoplus_{n=1}^{\infty} \tilde{A} \otimes_{\pi} A^{\otimes_{\pi} n} \\ a^0 da^1 \dots da^n &\leftrightarrow a^0 \otimes a^1 \otimes \dots \otimes a^n, \quad n \geq 0 \\ da^1 \dots da^n &\leftrightarrow 1 \otimes a^1 \otimes \dots \otimes a^n, \quad n \geq 1 \end{aligned} \quad (1.8)$$

where  $\tilde{A}$  is the algebra obtained from  $A$  by adjoining a unit and  $\otimes_{\pi}$  denotes the projective tensor product. The algebra  $\Omega A$  is naturally  $\mathbb{Z}/2\mathbb{Z}$ -graded by the decomposition into forms of even and odd degree, respectively. The basic operators on algebraic differential forms are the exterior derivative

$$d : \Omega^n A \longrightarrow \Omega^{n+1} A, \quad d(a^0 da^1 \dots da^n) = da^0 da^1 \dots da^n \quad (1.9)$$

and the Hochschild boundary operator

$$b : \Omega^n A \longrightarrow \Omega^{n-1} A, \quad b(\omega da) = (-1)^{|\omega|} [\omega, a] \quad (1.10)$$

They satisfy  $b^2 = d^2 = 0$ . The Karoubi operator

$$\kappa : \Omega^n A \longrightarrow \Omega^n A, \quad \kappa(a^0 da^1 \dots da^n) = (-1)^{n-1} da^n a^0 da^1 \dots da^{n-1} \quad (1.11)$$

can be expressed in terms of the two basic operators as  $\kappa = Id - (d \circ b + b \circ d)$ . Therefore it commutes with  $b$  and  $d$ . The Connes operator

$$B : \Omega^n A \longrightarrow \Omega^{n+1} A, \quad B = \sum_{i=0}^n \kappa^i \circ d \quad (1.12)$$

satisfies  $B^2 = 0$  and  $bB + Bb = 0$  and commutes with  $\kappa$  as well. The operators  $d$ ,  $b$  and  $B$  are of degree one with respect to the  $\mathbb{Z}/2\mathbb{Z}$ -grading of  $\Omega A$ .

Let  $\Omega A' = \prod \Omega^n A'$  be the dual space of bounded linear functionals on  $\Omega A$ . It is a  $\mathbb{Z}/2\mathbb{Z}$ -graded vector space. The operators  $d, b$  and  $B$  on  $\Omega A$  give rise to dual

operators on  $\Omega A'$  which will be denoted by the same letters. The periodic cyclic cochain complex of  $A$  is defined as

$$CC_{per}^*(A) = (\Phi = \prod \varphi_n : \Omega A = \bigoplus \Omega^n A \rightarrow \mathbb{C}, \varphi_n = 0, n \gg 0, b + B) \quad (1.13)$$

It is a  $\mathbb{Z}/2\mathbb{Z}$ -graded chain complex. Its cohomology  $HP^*(A)$  is the periodic cyclic cohomology of  $A$ .

Suppose that  $A$  is a Banach algebra. Then there are various cyclic (co)homology theories which take the topology of  $A$  more appropriately into account than the periodic cyclic theory. For a linear functional  $\Phi = \prod \varphi_n : \Omega A = \bigoplus \Omega^n A \rightarrow \mathbb{C}$  and a subset  $S \subset A$  put

$$\|\Phi\|_S = \sup_{n \in \mathbb{N}} \sup_{a^0, \dots, a^n \in S} c_n |\varphi_n(a^0, \dots, a^n)|, \quad c_{2n} = c_{2n+1} = \frac{1}{n!} \quad (1.14)$$

The entire and analytic cyclic cochain complexes of  $A$  are defined as

$$CC_\epsilon^*(A) = (\Phi : \Omega A \rightarrow \mathbb{C}, \|\Phi\|_S < \infty, S \subset A \text{ bounded}, b + B) \quad (1.15)$$

and

$$CC_{anal}^*(A) = (\Phi : \Omega A \rightarrow \mathbb{C}, \|\Phi\|_S < \infty, S \subset A \text{ compact}, b + B) \quad (1.16)$$

respectively. The cohomology groups of these  $\mathbb{Z}/2\mathbb{Z}$ -graded complexes are the entire cyclic cohomology  $HC_\epsilon^*(A)$  and the analytic cyclic cohomology groups  $HC_{anal}^*(A)$  of  $A$ , respectively. Finally there is the local cyclic cohomology theory  $HC_{loc}^*$  for Banach algebras. It is invariant under continuous homotopies which makes it particularly well behaved for  $C^*$ -algebras. Its definition will not be needed in the sequel.

The cyclic (co)homology theories discussed up to now appear as special cases of bi-variant homology theories  $HC_*^\beta(-, -)$ ,  $\beta = per, \epsilon, anal, loc$ . These theories are contravariant in the first and covariant in the second variable. The associated (co)homology theories are obtained by fixing one of the variables

$$HC_*^\beta(-) = HC_*^\beta(\mathbb{C}, -), \quad HC_\beta^*(-) = HC_*^\beta(-, \mathbb{C}) \quad (1.17)$$

The bivariant cohomology theories possess an associative and unital composition product

$$\circ : HC_*^\beta(A, B) \otimes HC_*^\beta(B, C) \longrightarrow HC_*^\beta(A, C) \quad (1.18)$$

and an associative exterior product

$$\times : HC_*^\beta(A, B) \otimes HC_*^\beta(C, D) \longrightarrow HC_*^\beta(A \otimes_\pi C, B \otimes_\pi D) \quad (1.19)$$

All cyclic theories listed up to now satisfy excision: short exact sequences of algebras with bounded linear section induce long exact sequences of bivariant cyclic groups in any of the two variables. There are obvious forgetful transformations

$$HC_*^{anal} \longrightarrow HC_*^\epsilon \longrightarrow HC_*^{per} \quad (1.20)$$

of homology and dual transformations

$$HC_{per}^* \longrightarrow HC_{\epsilon}^* \longrightarrow HC_{anal}^* \quad (1.21)$$

of cyclic cohomology theories. Moreover there is a natural forgetful transformation

$$HC_*^{anal}(-, -) \longrightarrow HC_*^{loc}(-, -) \quad (1.22)$$

of bifunctors which is multiplicative with respect to the composition and exterior products. All transformations are compatible with excision.

The interest in cyclic (co)homology theories originates from the existence of a natural Chern character

$$ch : K_* \longrightarrow HC_*^{loc} \rightarrow HC_*^{anal} \rightarrow HC_*^{\epsilon} \rightarrow HC_*^{per} \quad (1.23)$$

in operator  $K$ -theory which is compatible (up to universal constants) with exterior products and excision. The Chern character turns out to be a special case of a bivariant Chern-Connes character which will be discussed in 1.7.

## Characters of Fredholm modules

The need of character formulas for Fredholm modules was a basic motivation for A. Connes in his search for a noncommutative differential geometry [Co1]. Character formulas allow the explicit calculation of the index of a Fredholm module in exactly the same way as the Atiyah-Singer index formula leads to the calculation of indices of elliptic operators on compact manifolds. A character formula associates to a Fredholm module  $\mathcal{E}$  over  $A$  a cyclic cocycle  $\check{ch}(\mathcal{E})$  over  $A$  such that the index formula

$$\langle \check{ch}(\mathcal{E}), ch(e) \rangle = Index(\mathcal{E}_e)$$

holds for every  $K$ -theory class  $[e] \in K_0(A)$ . Here  $\mathcal{E}_e$  means the Fredholm module  $\mathcal{E}$  twisted by  $[e]$  and  $ch$  denotes the Chern character on  $K$ -theory with values in cyclic homology [Co1]. Contrary to the Chern character in  $K$ -theory there is no universal character formula working simultaneously for all Fredholm modules. All known explicit character formulas [Co1],[Co],[CM1], [Ni] work only under certain (sometimes rather restrictive) regularity assumptions on the involved operators. Moreover the cohomology class of the character cocycles changes definitely under homotopy. Therefore these characters cannot descend to  $K$ -homology. There is a character formula in analytic cyclic cohomology [Me] which works for all (bounded) modules without regularity assumptions but involves an infinite number of auxiliary choices. It descends in fact to  $K$ -homology but it is not known whether it can be extended to the bivariant setting. Finally there is a bivariant multiplicative Chern-Connes character on Kasparov's bivariant  $K$ -theory with values in bivariant local cyclic cohomology [Pu1]. Its existence is a consequence of the axiomatic characterization of  $KK$ -theory [Hi], but it is difficult to obtain explicit formulas for this abstract character because its construction makes use of excision in cyclic cohomology. Thus there are various partial solutions of the problem of finding a universal



character formula. The comparison of such partially defined characters can be quite tedious as can be seen in this paper.

We want to present two of the described characters and will study their relation in detail. The first one is A. Connes' character of a  $\theta$ -summable unbounded Fredholm module in entire cyclic cohomology [Co]. The other one is the abstract Chern-Connes character on  $K$ -homology with values in local cyclic cohomology [Pu1]. For simplicity we will treat only even Fredholm modules. All results of this paper hold for odd modules as well.

### Connes' character of a $\theta$ -summable unbounded Fredholm module

**Theorem 1.5.** [Co] *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . The functionals*

$$\begin{aligned} Ch_\epsilon(\mathcal{E}) &= (\varphi_{2n})_{n \in \mathbb{N}}, \quad \varphi_{2n}(a^0, \dots, a^{2n}) = \\ &= c_n \operatorname{Trace}' \left( \int_{1-i\infty}^{1+i\infty} F(z^2) a^0 [F(z^2), a^1] \dots [F(z^2), a^{2n}] e^{z^2} dz \right) \quad (1.24) \\ c_n &= \frac{1}{\sqrt{\pi}i} \left(-\frac{1}{4}\right)^n \frac{1}{n!}, \quad \operatorname{Trace}'(T) = \frac{1}{2} \operatorname{Trace}(T + \epsilon T \epsilon), \quad F(z) = \frac{\mathcal{D} + \epsilon z^{\frac{1}{2}}}{(\mathcal{D}^2 + z)^{\frac{1}{2}}} \end{aligned}$$

*define an entire cyclic cocycle [Co], pp.523,540 over the Banach algebra  $A$  (1.3).*

There are similar results for odd Fredholm modules [Co], [Co4]. A few remarks about the formula (1.24) are necessary.

- a) The expressions  $F(z) a^0 [F(z), a^1] \dots [F(z), a^{2n}], a^0, \dots, a^{2n} \in A, z \in \mathbb{C} \setminus \mathbb{R}_-$ , are usually not of trace class or in the domain of  $\operatorname{Trace}'$ .
- b) Only the integral described in (1.24) is in the domain of  $\operatorname{Trace}'$ , which consists of all bounded operators on  $\mathcal{H}$  whose even part is of trace class.
- c) The formula (1.24) does not explicitly involve the grading operator  $\epsilon$ .
- d) The holomorphic family  $F(z), z \in \mathbb{C} \setminus \mathbb{R}_-$  is constant modulo the compact operators:  $F(z) = \frac{\mathcal{D}}{(1+\mathcal{D}^2)^{\frac{1}{2}}} \bmod \mathcal{K}(\mathcal{H})$  and the latter expression does not involve the grading operator  $\epsilon$ .
- e) The grading operator  $\epsilon$  shows up only in the subdominant part  $\frac{\epsilon z^{\frac{1}{2}}}{(\mathcal{D}^2 + z)^{\frac{1}{2}}}$  of the operator family  $F(z)$ .

We comment on the grading in such detail because every even Fredholm module is homotopic to zero among ungraded modules.

**Theorem 1.6.** [Co], p.544 Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $e \in K_0(A)$ . Then the index formula

$$\langle Ch_\epsilon(\mathcal{E}), ch(e) \rangle = Index(\mathcal{E}_e) \quad (1.25)$$

holds where  $ch : K_*(A) \rightarrow HC_*^\epsilon(A)$  denotes the Chern character (1.23) in entire cyclic homology.

There is another well known entire cyclic character cocycle associated to an unbounded  $\theta$ -summable Fredholm module which was introduced by Jaffe, Lesniewski and Osterwalder [JLO] and turns out to be cohomologous to Connes' character cocycle as was shown in [Co4]. The JLO-cocycle is given by a simpler and more manageable formula than the cocycle of Connes. Connes' cocycle can be described in terms of universal algebras however which will enable us to relate it to the abstract bivariant characters. We note finally that there are character formulas for finitely summable unbounded Fredholm modules [Co1],[CM1]. They are cohomologous in entire cyclic cohomology to Connes character cocycle as well [CM1].

### The bivariant Chern-Connes character

**Theorem 1.7.** [Pu1], 6.3. *There exists a unique natural transformation*

$$ch_{biv} : KK_*(-, -) \longrightarrow HC_*^{loc}(-, -) \quad (1.26)$$

*of bifunctors on the category of separable  $C^*$ -algebras, which is multiplicative and not identically zero. It is compatible (up to universal constants) with exterior products and excision.*

The  $K$ -homology groups of a  $C^*$ -algebra can be described in terms of Kasparov theory as  $K^*(A) = KK_*(A, \mathbb{C})$ . Therefore one obtains as special case of (1.26) a character

$$\check{ch} : K^*(-) = KK_*(-, \mathbb{C}) \xrightarrow{ch_{biv}} HC_*^{loc}(-, \mathbb{C}) = HC_{loc}^*(-) \quad (1.27)$$

on  $K$ -homology with values in local cyclic cohomology.

According to Cuntz [Cu], [Cu1] the Chern-Connes character on (even)  $K$ -homology can be described in terms of universal algebras. We will recall a few facts from [Cu].

Let  $A$  be a  $C^*$ -algebra and let  $Q_{C^*}A = A * A$  be the free product of  $A$  with itself in the category of  $C^*$ -algebras. Denote by  $\gamma$  the involutive automorphism of  $QA$  switching the two copies of  $A$  and let  $\theta : A \rightarrow QA$  respectively  $\theta^\gamma : A \rightarrow QA$  be the canonical inclusions. There exists a natural extension

$$0 \rightarrow q_{C^*}A \rightarrow Q_{C^*}A \xrightarrow{id * id} A \rightarrow 0 \quad (1.28)$$

with the two natural multiplicative linear sections  $\theta$  and  $\theta^\gamma$ .

Consider the local cyclic (co)homology groups (1.17) associated to this extension. By excision the natural map

$$HC_*^{loc}(A, q_{C^*}A) \xrightarrow{\simeq} Ker ( HC_*^{loc}(A, Q_{C^*}A) \rightarrow HC_*^{loc}(A, A)) \quad (1.29)$$

is an isomorphism. We denote by

$$\iota_A^{loc} \in HC_*^{loc}(A, q_{C^*}A) \quad (1.30)$$

the natural element corresponding to the class

$$(\theta_* - \theta_*^\gamma) \in Ker ( HC_*^{loc}(A, Q_{C^*}A) \rightarrow HC_*^{loc}(A, A)) \quad (1.31)$$

under this isomorphism.

Let now  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, F)$  be an (even) bounded Fredholm module over the  $C^*$ -algebra  $A$  which satisfies the normalization conditions (1.4). There is a canonical pair of homomorphisms

$$\rho_0, \rho_1 : A \rightarrow \mathcal{L}(\mathcal{H}), \quad \rho_0(a) = \frac{1+\epsilon}{2} \rho(a) \frac{1+\epsilon}{2}, \quad \rho_1(a) = F \frac{1-\epsilon}{2} \rho(a) \frac{1-\epsilon}{2} F \quad (1.32)$$

These homomorphisms coincide modulo compact operators. In particular one obtains a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{K}(\mathcal{H}) & \longrightarrow & \mathcal{L}(\mathcal{H}) & \longrightarrow & \mathcal{Q}(\mathcal{H}) \rightarrow 0 \\ & & \psi \uparrow & & \uparrow \rho_0 * \rho_1 & & \uparrow \\ 0 & \rightarrow & q_{C^*}A & \longrightarrow & Q_{C^*}A & \longrightarrow & A \rightarrow 0 \end{array} \quad (1.33)$$

The map

$$\psi : q_{C^*}A \longrightarrow \mathcal{K}(\mathcal{H}) \quad (1.34)$$

is called the characteristic homomorphism associated to  $\mathcal{E}$ .

The local cyclic cohomology of the  $C^*$ -algebra of compact operators is given by

$$HC_{loc}^0(\mathcal{K}(\mathcal{H})) \simeq \mathbb{C} \quad (1.35)$$

The canonical generator

$$[\tau] \in HC_{loc}^0(\mathcal{K}(\mathcal{H})) = HC_0^{loc}(\mathcal{K}(\mathcal{H}), \mathbb{C}) \quad (1.36)$$

is cohomologous to the standard trace on the ideal  $\ell^1(\mathcal{H})$  of trace class operators.

We exhibit explicit cyclic cocycles which represent the generating class  $[\tau]$ . This is possible in local cyclic cohomology [Pu], section 7, but we prefer to use an analytic cyclic cocycle. We follow therefore [Me], section 3.4.

Let  $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$  be a sequence of finite rank projections in  $\mathcal{L}(\mathcal{H})$  such that  $\lim_{n \rightarrow \infty} Rank P_n = \infty$ ,  $\overline{\lim} (Rank P_n)^{\frac{1}{n}} < \infty$ . Then the cochain

$$\tau_{\mathcal{P}} = Trace - \partial(\mu_{\mathcal{P}}), \quad \mu_{\mathcal{P}} = (\mu_{\mathcal{P}}^{2n+1})_{n \in \mathbb{N}}, \quad (1.37)$$

$$n! \mu_{\mathcal{P}}^{2n+1}(a^0, \dots, a^{2n+1}) = \text{Trace}((1 - P_n) a^0 (1 - P_n) \dots (1 - P_n) a^{2n+1} (1 - P_n)) \quad (1.38)$$

defines an analytic cyclic cocycle on the  $C^*$ -algebra  $\mathcal{K}(\mathcal{H})$  of compact operators on  $\mathcal{H}$ . Its cohomology class does not depend on the choice of  $\mathcal{P}$ . The image of  $\tau_{\mathcal{P}}$  in local cyclic cohomology represents the fundamental class  $[\tau] \in HC_{loc}^0(\mathcal{K}(\mathcal{H}))$ .

With these notations at hand one has the following description of the Chern-Connes character on  $K$ -homology [Cu1], [Pu]:

Let  $\mathcal{E}$  be an even bounded Fredholm module over the  $C^*$ -algebra  $A$  and let  $[\mathcal{E}] \in K^0(A) = KK(A, \mathbb{C})$  be its  $K$ -homology class. Then

$$\check{ch}([\mathcal{E}]) = \iota_A^{loc} \circ \psi_* \circ [\tau] \in HC_0^{loc}(A, \mathbb{C}) = HC_{loc}^0(A) \quad (1.39)$$

In the same spirit it is possible to construct a bivariant character of unbounded Kasparov bimodules [BJ]. It depends only on the bivariant  $K$ -theory class of the considered bimodule. In [Ni] Nistor introduced a bivariant Chern-Connes character for finitely summable bimodules over pre- $C^*$ -algebras with values in bivariant periodic cyclic cohomology. It is compatible with the previously discussed bivariant character in the appropriate sense.

Our main goal will be the comparison of the two characters discussed above. A first step in this direction is the description of Connes' character 1.5 in the Cuntz-Quillen picture of cyclic cohomology. This is closely related to, but not identical with Connes' interpretation of his character in terms of traces on universal algebras in [Co].

### Connes' character in the Cuntz-Quillen picture

Cuntz and Quillen [CQ], [CQ1] gave a description of the periodic cyclic cochain complex of an algebra [Co1] in terms of the  $X$ -complex of the tensor algebra [Qu]. This description involves some extra structure on the tensor algebra which we recall first.

Let  $A$  be an algebra and let  $TA$  be the (nonunital) tensor algebra over  $A$ . Let  $\varrho : A \rightarrow TA$  be the canonical linear inclusion and let  $\omega : A \otimes A \rightarrow TA$ ,  $\omega(a, a') = \varrho(a a') - \varrho(a) \varrho(a')$  be its curvature [Qu]. Let

$$0 \rightarrow IA \rightarrow TA \xrightarrow{\pi} A \rightarrow 0 \quad (1.40)$$

be the natural extension of algebras with  $\varrho$  as natural linear section.

There is a natural linear isomorphism [CQ]

$$\begin{aligned}
\Omega^{ev} A &\simeq TA \\
a^0 da^1 \dots da^{2n} &\leftrightarrow \varrho(a^0) \omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n}) \\
\Omega^{odd} A &\simeq \Omega^1(TA)/[\Omega^1(TA), A] \\
a^0 da^1 \dots da^{2n+1} &\leftrightarrow \varrho(a^0) \omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n}) d\varrho(a^{2n+1})
\end{aligned} \tag{1.41}$$

Under this isomorphism the  $IA$ -adic filtration of  $TA$  corresponds to twice the degree filtration (Hodge filtration) of  $\Omega^{ev} A$ . The  $IA$ -adic completion of  $TA$  will be denoted by  $\widehat{TA}$ .

Now we recall Quillen's complex  $X^*(A)$ . It is the largest  $\mathbb{Z}/2\mathbb{Z}$ -graded subcomplex  $X^*(A) \subset CC_{per}^*(A)$  of the periodic cyclic cochain complex consisting of functionals on  $\Omega A$  which vanish on  $\Omega^n A$  for  $n > 1$ . Explicitly

$$X^0(A) = A', \quad X^1(A) = Ker(b : \Omega^1 A' \rightarrow \Omega^2 A') \tag{1.42}$$

The differentials are given by  $b : X^0(A) \rightarrow X^1(A)$ ,  $d : X^1(A) \rightarrow X^0(A)$ . Thus the even and odd cocycles in  $X^*(A)$  are given by the traces on  $A$  and the closed  $A$ -bimodule traces on  $\Omega^1 A$ , respectively.

The results of Cuntz and Quillen which we are going to use are the following

**Theorem 1.8.** [CQ1] *There is a natural diagram of chain homotopy equivalences*

$$CC_{per}^*(A) \xrightarrow{CC(\pi)} CC_{per}^*(\widehat{TA}) \hookrightarrow X^*(\widehat{TA}) \tag{1.43}$$

It is not true however that (1.41) extends to an isomorphism  $CC_{per}^*(A) \sim X^*(\widehat{TA})$  of chain complexes. Instead one has

**Theorem 1.9.** [CQ1] *Let  $A$  be an algebra.*

- a) *Let  $CC_{harm}^*(A) \subset CC_{per}^*(A)$  be the generalized 1-eigenspace of the Karoubi operator (1.11) acting on  $CC_{per}^*(A)$ . Then  $CC_{harm}^*(A)$  is a subcomplex and in fact a natural deformation retract of  $CC_{per}^*(A)$ .*
- b) *Let  $X^*(\widehat{TA})_{harm} \subset X^*(\widehat{TA})$  be the image of  $CC_{harm}^*(A)$  under the linear map (1.41). Then  $X^*(\widehat{TA})_{harm}$  is a subcomplex and in fact a natural deformation retract of  $X^*(\widehat{TA})$ .*
- c) *There is a natural isomorphism of the harmonic subcomplexes  $CC_{harm}^*(A) \simeq X^*(\widehat{TA})_{harm}$  which realizes the homotopy equivalence (1.43).*
- d) *The isomorphism of c) is given on forms of fixed degree by a scalar multiple of (1.41).*

In particular (1.41) maps even harmonic cocycles on the cyclic bicomplex bijectively to harmonic traces on the  $IA$ -adic completion  $\widehat{TA}$  of the tensor algebra  $TA$ .

Connes emphasizes in [Co] that his character cocycle  $Ch_\epsilon$  is normalized. This fact implies

**Lemma 1.10.** *The cocycle  $Ch_\epsilon$  is invariant under the Karoubi operator  $\kappa$  (1.11). In particular it is harmonic in the sense of 1.9.*

**Proof:** In fact

$$\begin{aligned} \kappa(Ch_\epsilon^{2n}(a^0, \dots, a^{2n})) &= Ch_\epsilon^{2n}(a^{2n}, a^0, \dots, a^{2n-1}) - Ch_\epsilon^{2n}(1, a^{2n}a^0, a^1, \dots, a^{2n-1}) \\ &= \tau(Fa^{2n}[F, a^0][F, a^1], \dots, [F, a^{2n-1}]) - \tau(F[F, a^{2n}a^0][F, a^1], \dots, [F, a^{2n-1}]) \\ &= -\tau(F[F, a^{2n}]a^0[F, a^1], \dots, [F, a^{2n-1}]) = \tau([F, a^{2n}]Fa^0[F, a^1], \dots, [F, a^{2n-1}]) \\ &= \tau(Fa^0[F, a^1], \dots, [F, a^{2n-1}][F, a^{2n}]) = Ch_\epsilon^{2n}(a^0, \dots, a^{2n}) \end{aligned}$$

□

The previous lemma allows to identify the cocycle  $Ch_\epsilon$  with a trace on the (suitably completed) tensor algebra  $TA$ . This makes it possible to give an alternative description of Connes' character cocycle. It should be noted that our description of the character cocycle in terms of traces on universal algebras is different from the one given by Connes in [Co].

**Lemma 1.11.** *Let  $\rho : A \rightarrow B$  be a homomorphism of algebras and let  $P = P^2 \in B$  be an idempotent element. Let  $\varphi : A \rightarrow B$ ,  $\varphi(a) = P\rho(a)P$  be the contraction with  $P$  and denote by  $T\varphi : TA \rightarrow B$  be the corresponding homomorphism of algebras. Then (in the notations of (1.41))*

$$T\varphi(\varrho(a^0)\omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n})) = (-1)^n P\rho(a^0)P[P, \rho(a^1)] \dots [P, \rho(a^{2n})] \quad (1.44)$$

**Proof:** By definition of the curvature of a linear map (we suppress  $\rho$  from the notation)

$$\begin{aligned} T\varphi(\omega(a, a')) &= Paa'P - (PaP)(Pa'P) \\ &= Pa(1 - P)a'P = P[a, 1 - P][a', P] = -P[P, a][P, a'] \end{aligned}$$

Since further  $P[P, a][P, a'] = P[P, a][P, a']P$  we deduce

$$\begin{aligned} T\varphi(\varrho(a^0)\omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n})) &= (Pa^0P)(-1)^n P[P, a^1] \dots [P, a^{2n}] \\ &= (-1)^n Pa^0P[P, a^1] \dots [P, a^{2n}] \end{aligned}$$

□

**Proposition 1.12.** Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . Let  $F(z) = \frac{\mathcal{D} + \epsilon z^{\frac{1}{2}}}{(\mathcal{D}^2 + z)^{\frac{1}{2}}} \in \mathcal{O}$  be the corresponding holomorphic family of operators where  $\mathcal{O}$  denotes the algebra of operator valued holomorphic functions on  $\mathbb{C} \setminus \mathbb{R}_-$ . Define linear maps  $\varphi_{0,1} : A \rightarrow \mathcal{O}$  by

$$\varphi_0(a) = \frac{1+F(z)}{2} \rho(a) \frac{1+F(z)}{2}, \quad \varphi_1(a) = \epsilon \frac{1-F(z)}{2} \rho(a) \frac{1-F(z)}{2} \epsilon \quad (1.45)$$

and let  $T\varphi_{0,1} : TA \rightarrow \mathcal{O}$  be the corresponding homomorphisms of algebras. Let finally

$$\tau(f) = \text{Trace}' \left( \frac{1}{\sqrt{\pi} i} \int_{1-i\infty}^{1+i\infty} f(z^2) e^{z^2} dz \right), \quad f \in \mathcal{O} \quad (1.46)$$

be the densely defined unbounded trace on  $\mathcal{O}$  introduced in (1.24). Then under the isomorphism of harmonic subcomplexes 1.9 the Connes character cocycle  $Ch_\epsilon(\mathcal{E}) \in CC_\epsilon(A)_{\text{harm}}^0$  corresponds to the harmonic trace  $T\varphi_0^*(\tau) - T\varphi_1^*(\tau) \in X^0(TA)_{\text{harm}}$ .

$$\begin{array}{ccc} CC_\epsilon(A)_{\text{harm}}^* & \simeq & X^*(TA)_{\text{harm}} \\ \in & & \ni \\ Ch_\epsilon(\mathcal{E}) & \leftrightarrow & T\varphi_0^*(\tau) - T\varphi_1^*(\tau) \end{array}$$

**Proof:** The linear map  $\varphi_0$  is given by contraction with the idempotent  $P_+ = \frac{1+F(z)}{2}$  and the linear map  $\varphi_1$  is the composition of the contraction with  $P_- = \frac{1-F(z)}{2}$  and the conjugation with  $\epsilon$ . Lemma 1.11 implies

$$\begin{aligned} T\varphi_0(a^0, \dots, a^{2n}) &= (-1)^n P_+ a^0 P_+ [P_+, a^1] \dots [P_+, a^{2n}] \\ &= \left(-\frac{1}{4}\right)^n P_+ a^0 P_+ [F, a^1] \dots [F, a^{2n}] \end{aligned}$$

whereas

$$T\varphi_1(a^0, \dots, a^{2n}) = \left(-\frac{1}{4}\right)^n \epsilon P_- a^0 P_- [F, a^1] \dots [F, a^{2n}] \epsilon$$

Therefore

$$\begin{aligned} (T\varphi_0^* - T\varphi_1^*)(\tau)(a^0, \dots, a^{2n}) &= \left(-\frac{1}{4}\right)^n \tau(P_+ a^0 P_+ [F, a^1] \dots [F, a^{2n}] - \epsilon P_- a^0 P_- [F, a^1] \dots [F, a^{2n}] \epsilon) \\ &= \left(-\frac{1}{4}\right)^n \tau((P_+ a^0 P_+ - P_- a^0 P_-)[F, a^1] \dots [F, a^{2n}]) \\ &= \left(-\frac{1}{4}\right)^n \tau(P_+ a^0 [F, a^1] \dots [F, a^{2n}] P_+ - P_- a^0 [F, a^1] \dots [F, a^{2n}] P_-) \\ &= \left(-\frac{1}{4}\right)^n \tau((P_+ - P_-) a^0 [F, a^1] \dots [F, a^{2n}]) = \left(-\frac{1}{4}\right)^n \tau(F a^0 [F, a^1] \dots [F, a^{2n}]) \end{aligned}$$

The isomorphism  $CC_\epsilon(A)_{\text{harm}}^* \simeq X^*(TA)_{\text{harm}}$  is given by (a suitable multiple) of (1.41) so that the conclusion follows.  $\square$

**Definition 1.13.** Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . Let  $\varphi_{0,1} : A \rightarrow \mathcal{O}$

$$\varphi_0(a) = \frac{1+F(z)}{2} a \frac{1+F(z)}{2}, \quad \varphi_1(a) = \epsilon \frac{1-F(z)}{2} a \frac{1-F(z)}{2} \epsilon \quad (1.47)$$

be the associated linear maps. Then the algebra homomorphism

$$\Phi = T\varphi_0 * T\varphi_1 : Q(TA) = TA * TA \longrightarrow \mathcal{O} \quad (1.48)$$

is called the characteristic homomorphism associated to  $\mathcal{E}$ .

## 2 Change of regularization in Connes' character formula

In this section we achieve two things. We show first that the algebraic description of the class of Connes' character in the Cuntz-Quillen picture, obtained in the previous section, leads to cocycles which satisfy all required continuity and boundedness properties. This is not very helpful yet, because the use of unbounded operators and the heat kernel regularization allows only a very limited use of algebraic operations, homotopies etc. These are indispensable however in order to achieve the transgression of Connes' character to the character of a bounded Fredholm module. The second issue of this section is therefore a change of regularization in the character formulas.

It is well known that the canonical trace  $\tau$  on the algebra  $\ell^1(\mathcal{H})$  of trace class operators on Hilbert space does not extend to a trace on the  $C^*$ -closure  $\mathcal{K}(\mathcal{H})$  of all compact operators. The induced map  $HC_{loc}^*(\mathcal{K}(\mathcal{H})) \xrightarrow{\sim} HC_{loc}^*(\ell^1(\mathcal{H}))$  on local cyclic cohomology is however an isomorphism [Pu2].

Therefore the canonical trace, viewed as a local cyclic cocycle, is cohomologous to a cocycle  $\tau_{\mathcal{P}}$  which extends to all of  $\mathcal{K}(\mathcal{H})$ . Such a cocycle depends on the choice of a bounded approximate unit  $\mathcal{P}$  in  $\mathcal{K}(\mathcal{H})$  and may be analytic [Me].

Connes constructs his character cocycle in terms of a superalgebra  $\tilde{\mathcal{L}}$  of operator-valued distributions and a canonical odd trace  $\tilde{\tau}$  on it. We will replace in all formulas his trace by a cyclic cocycle  $\tilde{\tau}_{\mathcal{P}}$  on  $\tilde{\mathcal{L}}$  which resembles the regularized cocycles  $\tau_{\mathcal{P}}$  on  $\mathcal{K}(\mathcal{H})$ . This will enable us to perform the crucial transgression in the next section.

Almost the whole section consists of estimates near zero of the Schatten norms of operator-valued distributions on the real halfline and of pointwise estimates of the operator norm of their Laplace transforms. We recommend some familiarity with the calculations in [Co] in order to digest the following pages.

### Operator valued distributions and the Laplace transform

It turns out to be useful to view the holomorphic families of operators which occur in Connes' character formula 1.5 as the Laplace transforms of certain operator valued



distributions. This motivated Connes to introduce the following convolution algebra of distributions on the positive real halfline.

**Definition 2.1.** [Co], p.531.

Let  $\mathcal{H}$  be a Hilbert space. The convolution algebra  $\mathcal{L}$  of operator valued distributions consists of all tempered distributions  $T$  on the real line satisfying

- a)  $Supp(T) \subset [0, \infty[$
- b) There exists a holomorphic operator valued function  $t$  on some open cone  $\bigcup_{s>0} sB(1, r)$ ,  $B(1, r) = \{z \in \mathbb{C}, |z - 1| < 1\}$ ,  $0 < r < 1$  in the open right halfplane such that  $T$  coincides with  $t$  on  $]0, \infty[$
- c) the function

$$h(p) = \sup_{z \in \frac{1}{p}B(1, r)} \|t(z)\|_p, \quad p \in ]1, \infty[$$

is of polynomial growth where  $\| - \|_p$  denotes the Schatten  $p$ -norm.

It is a nontrivial fact that  $\mathcal{L}$  is indeed an algebra (see [Co], p.533). Connes introduces also the following superalgebra of distributions.

**Definition 2.2.** [Co], p.534.

Let  $\mathcal{H}$  be a Hilbert space and let  $\mathcal{L}$  be the associated algebra 2.1 of distributions. Denote by  $\tilde{\mathcal{L}}$  the quadratic Galois extension of  $\mathcal{L}$  obtained by adjoining a square root  $\lambda^{\frac{1}{2}}$  of the distribution  $\lambda = \delta'_0$ . Then  $\lambda^{\frac{1}{2}}$  is a central element in  $\tilde{\mathcal{L}}$  and  $Gal(\tilde{\mathcal{L}}, \mathcal{L}) = \mathbb{Z}/2\mathbb{Z}$ .

The Laplace transform is a homomorphism  $L$  from the convolution algebra  $\mathcal{L}$  of operator valued distributions to the algebra of operator valued holomorphic functions on the open right half plane  $\{z \in \mathbb{C}, Re(z) > 0\}$ . It is given by the formula

$$L(T)(z) = \int_0^\infty T(s) e^{-sz} ds \quad (2.1)$$

for all  $T \in \mathcal{L}$ . We extend it to a homomorphism  $\tilde{L}$  on  $\tilde{\mathcal{L}}$  by sending  $\lambda^{\frac{1}{2}} \in \tilde{\mathcal{L}}$  to the branch  $z^{\frac{1}{2}} \in \mathcal{O}$  of the square root function which takes the value  $+1$  at  $1$ . The Laplace transform maps  $\mathcal{L}$  isomorphically onto its picture. An explicit formula for the value of the inverse Laplace transform  $\tilde{L}^{-1} : \tilde{L}(\tilde{\mathcal{L}}) \rightarrow \tilde{\mathcal{L}}$  at  $s > 0$  is given on the odd part of  $\tilde{\mathcal{L}}$  by

$$T(s) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} S(z) z^{-\frac{1}{2}} e^{sz} dz = \int_\Gamma S(z^2) d\mu_s(z) \quad (2.2)$$

where  $S = \tilde{L}(\lambda^{\frac{1}{2}} T)$ ,  $T \in \mathcal{L}$ , and

$$d\mu_s(z) = \frac{1}{\pi i} e^{sz^2} dz \quad (2.3)$$

is a suitable Gaussian measure. The integration in the last expression is carried out over the oriented curve

$$\Gamma = \{z \in \mathbb{C}, \arg(z-1) = \pm \frac{\pi}{3}\} \quad (2.4)$$

going from the lower to the upper halfplane. The equality of the two integrals follows from the Cauchy integral formula by a reasoning similar to that in [Co], pp. 542-543.

We will suppress the symbol  $\tilde{L}$  for the Laplace transform from the notation. If  $T \in \tilde{\mathcal{L}}$  is an operator valued distribution, then its Laplace transform will be denoted by  $T(z)$ .

**Definition and Lemma 2.3.** [Co], p.534.

Let  $\mathcal{H}$  be a Hilbert space and let  $\tilde{\mathcal{L}}$  (see 2.1, 2.2) be the associated superalgebra of distributions. Then the functional

$$\tilde{\tau}(T) = \text{Trace}(T_-(1)), \quad T = T_+ + \lambda^{\frac{1}{2}} T_- \in \tilde{\mathcal{L}}, \quad T_{\pm} \in \mathcal{L} \quad (2.5)$$

defines an (odd) trace on the convolution algebra  $\tilde{\mathcal{L}}$ . It corresponds under the Laplace transform to a multiple of the trace functional introduced in (1.24).

**Definition 2.4.** Let  $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$  be a sequence of finite rank projections in  $\mathcal{L}(\mathcal{H})$  such that  $\lim_{n \rightarrow \infty} \text{Rank } P_n = \infty$ ,  $\overline{\lim} (\text{Rank } P_n)^{\frac{1}{n}} < \infty$ . Then the functional cochain

$$\tilde{\tau}_{\mathcal{P}} = \tilde{\tau} - \partial(\tilde{\mu}_{\mathcal{P}}), \quad \tilde{\mu}_{\mathcal{P}} = (\mu_{\mathcal{P}}^{2n+1})_{n \in \mathbb{N}}, \quad (2.6)$$

$$n! \tilde{\mu}_{\mathcal{P}}^{2n+1}(T^0, \dots, T^{2n+1}) = \tilde{\tau}((1 - P_n) T^0 (1 - P_n) \dots (1 - P_n) T^{2n+1} (1 - P_n))$$

is a cocycle on the chain complex  $CC^0(\tilde{\mathcal{L}})$  which is cohomologous to the canonical trace  $\tilde{\tau}$  on  $\tilde{\mathcal{L}}$

Connes associates to every unbounded  $\theta$ -summable Fredholm module a couple of operator valued distributions. These are used to establish the continuity of the character cocycle (1.24).

**Definition and Lemma 2.5.** [Co], p.535. Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ .

i) There exist operator-valued distributions [Co], p.535,

$$F = \mathcal{D}N + \lambda^{\frac{1}{2}} \epsilon N \in \tilde{\mathcal{L}}, \quad N = \frac{1}{\sqrt{\pi s}} e^{-s\mathcal{D}^2} \in \mathcal{L}, \quad \lambda^{\frac{1}{2}} * \lambda^{\frac{1}{2}} = \delta'_0 \quad (2.7)$$

satisfying

$$F^2 = \delta_0 = 1 \quad (2.8)$$

ii) The Laplace transform of the distribution in i) is given by

$$\tilde{L}(F)(z) = \frac{\mathcal{D} + z^{\frac{1}{2}} \epsilon}{(\mathcal{D}^2 + z)^{\frac{1}{2}}} \quad (2.9)$$

**Definition 2.6.** We denote by  $\varphi_0, \varphi_1 : A \rightarrow \tilde{\mathcal{L}}$  the linear maps

$$\varphi_0(a) = \frac{1+F}{2} \rho(a) \frac{1+F}{2}, \quad \varphi_1(a) = \epsilon \frac{1-F}{2} \rho(a) \frac{1-F}{2} \epsilon \quad (2.10)$$

whose Laplace transform yields the maps introduced in 1.13. We call the induced algebra homomorphism

$$\Phi = T\varphi_0 * T\varphi_1 : Q(TA) \rightarrow \tilde{\mathcal{L}} \quad (2.11)$$

still the characteristic morphism associated to  $\mathcal{E}$ . Its Laplace transform is the morphism of 1.13.

### Majorizing functions

We recall a few facts from [Co].

**Definition 2.7.** [Co], page 537.

Let  $B(1, r) = \{z \in \mathbb{C}, |z - 1| < r\}$ . Then  $f \in L^1([0, 1])$  is a majorizing function for  $T \in \mathcal{L}$  with respect to  $B(1, r)$  if

$$\text{a) } \sup_{z \in \frac{1}{p}B(1, r)} \|T(z)\|_p \leq f\left(\frac{1}{p}\right), \quad \forall p \in [1, \infty]$$

$$\text{b) } T(\Phi) = \int_0^\infty T(s)\Phi(s)ds, \quad \forall \Phi \in \mathcal{S}(\mathbb{R})$$

We write  $T \prec_r f$  in this case.

Remark: Not every distribution in  $\mathcal{L}$  possesses a majorizing function.

**Lemma 2.8.** [Co]

Let  $f \in L^1([0, 1])$  be a majorizing function for  $T \in \mathcal{L}$  with respect to  $B(1, r)$  such that  $s^{-k} f \in L^1([0, 1])$ . Then

$$\frac{\partial^k T}{\partial s^k} \prec_{\frac{r}{2}} \left(\frac{2}{r}\right)^k k! (s^{-k} f) \quad (2.12)$$

**Lemma 2.9.** [Co], Lemma 1, page 537.

$$T_0 \prec_r f_0, \quad T_1 \prec_r f_1 \implies T_0 * T_1 \prec_r 2f_0 * f_1, \quad \forall r < 1 \quad (2.13)$$

**Lemma 2.10.** Define smooth functions  $f_n$  on  $\mathbb{R}_+$  for  $n \geq 1$  by

$$f_1(s) = \frac{1}{\sqrt{\pi}s}, \quad f_{n+m} = f_n * f_m$$

where the product is given by convolution on the additive semigroup  $(\mathbb{R}_+, +)$ . Then

$$f_{2n}(s) = \frac{s^{n-1}}{(n-1)!}, \quad f_{2n+1}(s) = \frac{1}{\sqrt{\pi}} 4^n \frac{(n)!}{(2n)!} s^{n-\frac{1}{2}} \quad (2.14)$$

**Proof:** Take the Laplace transforms  $\widehat{f}(s) = \int_0^\infty f(t)e^{-st}dt$  of the functions in question.  $\square$

**Lemma 2.11.** *Let  $T \in \mathcal{L}$  be an operator valued distribution, let  $0 < r < 1$  and let  $n > 0$  be an integer. Then*

$$T \underset{r}{\prec} f_{n+2k}(s) \Rightarrow \frac{\partial^k T}{\partial s^k} \underset{\frac{r}{2}}{\prec} C(r)^k f_n(s) \quad (2.15)$$

This is clear from 2.8, 2.9 and 2.10.

### Estimates of Schatten norms of operator valued distributions

We fix now an even unbounded  $\theta$ -summable Fredholm module  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  over  $A$  and recall that  $A$  is supposed to be a Banach algebra with respect to the norm

$$\|a\|_A = \|\rho(a)\|_{\mathcal{L}(\mathcal{H})} + \|[\mathcal{D}, \rho(a)]\|_{\mathcal{L}(\mathcal{H})}, \quad a \in \mathcal{A}$$

We will omit  $\rho$  from the notation if no confusion is likely to arise.

Following Connes we consider the associated operator-valued distributions (2.7).

**Lemma 2.12.** *[Co], pp.535, 537. Let  $\mathcal{E} = (\mathcal{H}, \rho, \mathcal{D}, \epsilon)$  be an even unbounded  $\theta$ -summable unbounded Fredholm module over  $A$  and let  $N, \mathcal{D}N$  be the associated distributions. Let  $P$  be any spectral projection associated to  $\mathcal{D}^2$  and put  $\eta = \text{Trace}(P e^{-\frac{\mathcal{D}^2}{4}})$ .*

a)

$$P\mathcal{D}N = PS\delta_0 + (1 - \lambda)PT, \quad S \in \mathcal{L}(\mathcal{H}), T \in \mathcal{L} \quad (2.16)$$

with

$$PT \underset{\frac{1}{4}}{\prec} C_1(\mathcal{E})\eta^s f_1(s), \quad PN \underset{\frac{1}{4}}{\prec} C_2(\mathcal{E})\eta^s f_1(s) \quad (2.17)$$

([Co], Lemma 2 b) p. 535.

b) For all  $a \in A$  one has

$$P[F, a]P = P[F, a]_+P + \lambda^{\frac{1}{2}} P[F, a]_-P, \quad [F, a]_{\pm} \in \mathcal{L}$$

with

$$P[F, a]_+P \underset{\frac{1}{4}}{\prec} C_3(\mathcal{E})\eta^s f_1(s) \|a\|_A, \quad P[F, a]_-P \underset{\frac{1}{4}}{\prec} C_4(\mathcal{E})\eta^s f_2(s) \|a\|_A \quad (2.18)$$

c) Suppose that  $\text{Im}(P) \cap \text{Ker}(\mathcal{D}) = 0$  and that  $\eta < 1$ . Then

$$PT \underset{\frac{1}{4}}{\prec} C_5(\mathcal{E})(-4 \log \eta)^{-1} \eta^s f_1(s). \quad (2.19)$$

**Proof:** This is a mild generalization of [Co], Lemma 2, p.537. The proof of a) and b) is exactly the same as in loc. cit. Concerning c) we observe (in the notations of [Co]) that

$$PT = \frac{1}{\pi} \int_0^\infty P \mathcal{D}(\mathcal{D}^2 + 1 + \rho)^{-1} (P e^{-s(\mathcal{D}^2 + \rho)}) \rho^{-\frac{1}{2}} d\rho$$

Let  $\lambda_1 > 0$  be the smallest eigenvalue of the restriction of  $\mathcal{D}^2$  to the image of  $P$ . Then

$$\| P \mathcal{D}(\mathcal{D}^2 + 1 + \rho)^{-1} \| \leq \lambda_1^{-1} \leq (-4 \log \eta)^{-1}$$

because  $e^{-\frac{\lambda_1}{4}} < \text{Trace}(P e^{-\frac{\mathcal{D}^2}{4}}) = \eta$ . On the other hand one finds for the Schatten norm

$$\left\| \frac{1}{\pi} \int_0^\infty (P e^{-s(\mathcal{D}^2 + \rho)}) \rho^{-\frac{1}{2}} d\rho \right\|_{\frac{1}{s}} \leq \eta^s \frac{1}{\pi} \int_0^\infty e^{-s\rho} \rho^{-\frac{1}{2}} d\rho = \eta^s f_1(s)$$

which suffices to establish our claim.  $\square$

Let  $TA$  be the (nonunital) tensor algebra over  $A$ . Let  $\varrho : A \rightarrow TA$  be the canonical linear inclusion and let  $\omega : A \otimes A \rightarrow TA$ ,  $\omega(a, a') = \varrho(a a') - \varrho(a) \varrho(a')$  be its curvature [Qu]. Recall that there is a natural linear isomorphism

$$\Omega^{ev} A \quad \simeq \quad TA$$

$$a^0 da^1 \dots da^{2n} \leftrightarrow \varrho(a^0) \omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n})$$

**Remark 2.13.** *In the sequel we only will prove estimates for tensors of the form  $\varrho \omega^n$  and not of the form  $\omega^n$ , which correspond to exact forms under (1.41). This is purely a matter of notational convenience. All estimates of this section are valid for tensors of the latter form as well. For them the proofs are usually even simpler.*

**Lemma 2.14.** *Let  $S \subset A$  be a bounded set and put  $\| S \| = \sup_{a \in S} \| a \|$ . Let  $\alpha \in TA$  be an element of the form  $\alpha = \varrho \omega^n(a^0, \dots, a^{2n})$ ,  $a^0, \dots, a^{2n} \in S$ ,  $n > 0$ . Let  $T\varphi_0, T\varphi_1 : TA \rightarrow \tilde{\mathcal{L}}$  be the homomorphisms introduced in 2.6. Then*

$$T\varphi_i(\alpha) = T\varphi_i(\alpha)_+ + \lambda^{\frac{1}{2}} T\varphi_i(\alpha)_-, \quad T\varphi_i(\alpha)_\pm \in \mathcal{L}, i = 0, 1,$$

with

$$T\varphi_i(\alpha)_+ \prec_{\frac{1}{16}} (C_6(\mathcal{E}) \| S \|^ {2n+1} f_{2n-1}(s), \quad T\varphi_i(\alpha)_- \prec_{\frac{1}{16}} (C_7(\mathcal{E}) \| S \|^ {2n+1} f_{2n}(s)) \quad (2.20)$$

**Proof:** This is essentially Lemma 3 on page 538 of [Co]. As similar arguments will be used frequently in this section we will give all necessary details.

The calculations in 1.12 show that

$$T\varphi_0(\varrho(a^0) \omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n})) = (-\frac{1}{4})^n P_+ a^0 P_+ [F, a^1] \dots [F, a^{2n}]$$

$$T\varphi_1(\varrho(a^0)\omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n})) = (-\frac{1}{4})^n \epsilon P_- a^0 P_- [F, a^1] \dots [F, a^{2n}] \epsilon$$

Now

$$[F, a^1] \dots [F, a^{2n}] = T_+ + \lambda^{\frac{1}{2}} T_-, \quad T_+, T_- \in \mathcal{L},$$

with

$$T_+ = \sum_{j=0}^n \sum_{J' \subset \{1, \dots, 2n\}, |J'|=2j} \lambda^j T_{J'}, \quad T_- = \sum_{j=0}^{n-1} \sum_{J'' \subset \{1, \dots, 2n\}, |J''|=2j+1} \lambda^j T_{J''}$$

where for  $J \subset \{1, \dots, 2n\}$  one has  $T_J = S_1 \dots S_{2n}$  with  $S_i = [F, a^i]_-$  if  $i \in J$  and  $S_i = [F, a^i]_+$  otherwise. From this one derives with the help of 2.8, 2.9, 2.11 and 2.12 the estimates

$$T_{J'} \prec_{\frac{1}{4}} 2^{2n-1} C_8^{2n-2j} C_9^{2j} \|a^1\|_A \dots \|a^{2n}\|_A f_{2n+2j}(s)$$

$$\leq (2C_8 + 2C_9)^{2n} \|S\|^{2n} f_{2n+2j}(s)$$

and

$$\lambda^j T_{J'} \prec_{\frac{1}{8}} C_{10}^{2n} \|S\|^{2n} f_{2n}(s)$$

Thus

$$T_+(a^1, \dots, a^{2n}) \prec_{\frac{1}{8}} (C_{11}(\mathcal{E}) \|S\|)^{2n} f_{2n}(s)$$

A similar calculation shows

$$T_-(a^1, \dots, a^{2n}) \prec_{\frac{1}{8}} (C_{12}(\mathcal{E}) \|S\|)^{2n} f_{2n+1}(s)$$

We consider now the distributions  $P_{\pm} a^0 P_{\pm}$ . One finds from 2.5, 2.6, and 2.12

$$P_{\pm} a P_{\pm} = P_{\pm} a - P_{\pm} [P_{\pm}, a] = S^{\pm}(a) \delta_0 + T_1^{\pm}(a) + \lambda^{\frac{1}{2}} T_2^{\pm}(a) + \lambda T_3^{\pm}(a) \quad (2.21)$$

$$\|S(P_{\pm})(a)\|_{\mathcal{L}(\mathcal{H})} \leq C_{13} \|a\|_A, \quad T_i^{\pm}(a) \prec_{\frac{1}{8}} C_{14} \|a\|_A f_1(s), \quad i = 1, 2, 3 \quad (2.22)$$

Altogether

$$\begin{aligned} & T\varphi_0(\varrho(a^0)\omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n})) = \\ & = (S^+ \delta_0 + T_1^+ + \lambda^{\frac{1}{2}} T_2^+ + \lambda T_3^+)(a^0)(T_+(a^1, \dots, a^{2n}) + \lambda^{\frac{1}{2}} T_-(a^1, \dots, a^{2n})) \\ & = T\varphi_0(\varrho\omega^n)_+ + \lambda^{\frac{1}{2}} T\varphi_0(\varrho\omega^n)_- \end{aligned}$$

where according to 2.9, 2.10, 2.11,

$$T\varphi_0(\varrho\omega^n)_+ \prec_{\frac{1}{16}} (C_{15}(\mathcal{E}) \|S\|)^{2n+1} f_{2n-1}(s)$$

$$T\varphi_0(\varrho\omega^n)_- \prec_{\frac{1}{16}} (C_{16}(\mathcal{E}) \|S\|)^{2n+1} f_{2n}(s)$$

A similar calculation applies for  $T\varphi_1$ . □

Let  $A$  be an algebra and let  $QA = A * A$  be the free product of  $A$  with itself. Denote by  $\gamma$  the involutive automorphism switching the two copies of  $A$  and let  $\theta : A \rightarrow QA$  be the inclusion of the first copy of  $A$  into the free product. For  $a \in A$  denote by  $pa$  and  $qa$  the even and the odd parts of  $\theta(a)$  with respect to the involution  $\gamma$ . Recall that there is a natural linear isomorphism [CQ], 1.3

$$\begin{aligned} \Omega A &\xrightarrow{\cong} QA \\ a^0 da^1 \dots da^n &\leftrightarrow pa^0 qa^1 \dots qa^n \end{aligned} \quad (2.23)$$

**Remark 2.15.** *For simplicity we will only prove estimates for elements in a free product of the form  $pq^n$  and not of the form  $q^n$  corresponding to exact forms under (2.23). All estimates of this section are however valid for elements of the latter form and for them the proofs are even simpler.*

**Lemma 2.16.** *Let  $\Phi : Q(TA) \rightarrow \tilde{\mathcal{L}}$  be the characteristic homomorphism (2.10). Let  $S \subset A$  be a bounded subset of  $A$ .*

a) *For every element  $\alpha \in Q(TA)$  of the form*

$$\alpha = p(\varrho\omega^{n_0})q(\varrho\omega^{n_1}) \dots q(\varrho\omega^{n_k})(a^0, \dots, a^{k+2l}), \quad (2.24)$$

$$a^0, \dots, a^{k+2l} \in S, l = (n_0 + \dots + n_k) > 0, k \geq 0, m = \#\{i, n_i > 0\} \leq l$$

*one has*

$$\Phi(\alpha) = \sum_{i=0}^2 \lambda^{\frac{i}{2}} \Phi_i(\alpha), \Phi_i(\alpha) \in \mathcal{L}, \quad (2.25)$$

*with*

$$\begin{aligned} \Phi_i(\alpha) &\prec_{\frac{1}{64}} (C_{17}(\mathcal{E}) \| S \|)^{k+2l+1} f_{2l-m}(s), \quad i = 0, 1 \\ \Phi_2(\alpha) &\prec_{\frac{1}{64}} (C_{18}(\mathcal{E}) \| S \|)^{k+2l+1} f_{2l-m+1}(s) \end{aligned} \quad (2.26)$$

b) *For every element*

$$\alpha = p(\varrho)q(\varrho)^k(a^0, \dots, a^k), \quad (2.27)$$

*$a^0, \dots, a^k \in S, k \geq 0$ , one has*

$$\Phi(\alpha) = S(\alpha) \delta_0 + \sum_{j=0}^3 \lambda^{\frac{j}{2}} \Phi_j(\alpha), \Phi_j(\alpha) \in \mathcal{L}, \quad (2.28)$$

*with*

$$\begin{aligned} \Phi_j(\alpha) &\prec_{\frac{1}{64}} (C_{19}(\mathcal{E}) \| S \|)^{k+1} f_1(s), \quad j = 0, 1, 2 \\ \Phi_3(\alpha) &\prec_{\frac{1}{64}} (C_{20}(\mathcal{E}) \| S \|)^{k+1} f_2(s) \end{aligned} \quad (2.29)$$

*The term  $S(\alpha)$  occurs only if  $k = 0$  and then  $\| S(\alpha) \| \leq C_{21}(\mathcal{E}) \| S \|$ .*

**Proof:**

If  $\alpha = q(\varrho\omega^n(a^0, \dots, a^{2n}), a^0, \dots, a^{2n}) \in S$ ,  $n > 0$ , then

$$\Phi(\alpha)_+ = \frac{1}{2}(T\varphi_0(\alpha)_+ - T\varphi_1(\alpha)_+) \prec_{\frac{1}{16}} (C_{22}(\mathcal{E}) \| S \|)^{2n+1} f_{2n-1}(s)$$

$$\Phi(\alpha)_- = \frac{1}{2}(T\varphi_0(\alpha)_- - T\varphi_1(\alpha)_-) \prec_{\frac{1}{16}} (C_{23}(\mathcal{E}) \| S \|)^{2n+1} f_{2n}(s)$$

by lemma 2.14. A similar estimate holds for  $\alpha = p(\varrho\omega^n(a^0, \dots, a^{2n}), a^0, \dots, a^{2n}) \in S$ ,  $n > 0$ . If  $\alpha = q(\varrho(a))$ ,  $a \in S$ , we find the following.

$$\begin{aligned} \Phi(q\varrho(a)) &= \frac{1}{2}(P_+aP_+ - \epsilon P_-aP_- \epsilon) \\ &= \frac{1}{2}(P_+\epsilon a(\epsilon P_+ - P_- \epsilon) + (P_+\epsilon - \epsilon P_-)aP_- \epsilon) = \\ &= \frac{1}{2} \left( \frac{1+F}{2} \epsilon a \lambda^{\frac{1}{2}} N + \lambda^{\frac{1}{2}} N a \frac{1-F}{2} \epsilon \right) \\ &= \frac{1}{4} \lambda^{\frac{1}{2}} \epsilon ((Na + aN) - N[\mathcal{D}, a]N) \end{aligned} \quad (2.30)$$

Thus

$$\Phi(q\varrho(a)) = \lambda^{\frac{1}{2}} T_0(a), \quad T_0(a) \prec_{\frac{1}{4}} C_{24} \| a \|_A f_1(s) \quad (2.31)$$

by 2.9 and 2.12. A calculation using 2.9 and 2.12 again shows that

$$\Phi(p(\varrho(a))) = S(a) \delta_0 + \sum_{j=0}^3 \lambda^{\frac{j}{2}} \Phi_j, \quad (2.32)$$

$$\| S \| \leq C_{25} \| S \|, \quad \Phi_j \prec_{\frac{1}{32}} C_{25} \| S \| f_1(s), \quad 0 \leq j \leq 2, \quad \Phi_3 \prec_{\frac{1}{32}} C_{26} \| S \| f_3(s) \quad (2.33)$$

Products of the form  $p(\varrho\omega^{n_0})q(\varrho\omega^{n_1}) \dots q(\varrho\omega^{n_k})$  of the terms considered up to now can be analyzed in the same way as in the proof of lemma 2.14.  $\square$

**Lemma 2.17.** *Let  $P$  be a spectral projection associated to  $\mathcal{D}^2$  and put  $\eta = \text{Trace}(P e^{-\frac{\mathcal{D}^2}{4}})$ . Define linear maps  $(T\varphi_i)^P : TA \rightarrow \tilde{\mathcal{L}}$  by*

$$(T\varphi_0)^P(\varrho\omega^n(a^0, \dots, a^{2n})) = (-\frac{1}{4})^n P(P_+a^0P_+)P[F, a^1]P \dots P[F, a^{2n}]P$$

$$(T\varphi_1)^P(\varrho\omega^n(a^0, \dots, a^{2n})) = (-\frac{1}{4})^n \epsilon P(P_-a^0P_-)P[F, a^1]P \dots P[F, a^{2n}]P\epsilon$$

and put  $p^P = \frac{1}{2}((T\varphi_0)^P + (T\varphi_1)^P)$ ,  $q^P = \frac{1}{2}((T\varphi_0)^P - (T\varphi_1)^P)$ . Let finally  $\Phi^P : Q(TA) \rightarrow \tilde{\mathcal{L}}$  be the linear map defined by

$$\Phi^P(p(\varrho\omega^{n_0})q(\varrho\omega^{n_1}) \dots q(\varrho\omega^{n_k})) = p^P(\varrho\omega^{n_0})q^P(\varrho\omega^{n_1}) \dots q^P(\varrho\omega^{n_k})$$



Then the statement of lemma 2.16 holds for  $\Phi^P$  (instead of  $\Phi$ ) with the following two modifications: the constants  $C_{17}(\mathcal{E}), \dots, C_{21}(\mathcal{E})$  have possibly to be changed and all majorizing functions  $\Phi_i, \Phi_j$  have to be replaced by  $\eta^s \Phi_i, \eta^s \Phi_j$ .

Suppose moreover that  $\text{Im}(P) \cap \ker(\mathcal{D}) = 0$  and that  $\eta < 1$ . Let  $S \subset A$  be a bounded subset of  $A$ . Then

a) For all elements  $\alpha \in q(TA)$  of the form (2.24) one has

$$\Phi^P(\alpha) = \sum_{i=0}^3 \lambda^{\frac{i}{2}} \Phi_i^P(\alpha), \quad \Phi_i^P(\alpha) \in \mathcal{L}, \quad (2.34)$$

with

$$\begin{aligned} \Phi_i^P(\alpha) &\prec_{\frac{1}{64}} (C_{27}(\mathcal{E}) \parallel S \parallel)^{k+2l+1} f_{2l-m+i}(s), & i = 0, 1 \\ \Phi_i^P(\alpha) &\prec_{\frac{1}{64}} (C_{28}(\mathcal{E}) \parallel S \parallel)^{k+2l+1} (-4 \log \eta)^{-1} f_{2l-m-1+i}(s) & i = 2, 3 \end{aligned} \quad (2.35)$$

b) For all elements  $\alpha \in q(TA)$  of the form (2.27) one has

$$\Phi^P(\alpha) = \sum_{j=0}^3 \lambda^{\frac{j}{2}} \Phi_j^P(\alpha), \quad \Phi_j^P(\alpha) \in \mathcal{L}, \quad (2.36)$$

with

$$\begin{aligned} \Phi_j^P(\alpha) &\prec_{\frac{1}{64}} (C_{29}(\mathcal{E}) \parallel S \parallel)^{k+1} f_1(s), & j = 0, 1 \\ \Phi_2^P(\alpha) &\prec_{\frac{1}{64}} (C_{30}(\mathcal{E}) \parallel S \parallel)^{k+1} ((-4 \log \eta)^{-1} f_1(s) + f_2(s)), \\ \Phi_3^P(\alpha) &\prec_{\frac{1}{64}} (C_{31}(\mathcal{E}) \parallel S \parallel)^{k+1} (-4 \log \eta)^{-1} f_2(s) \end{aligned} \quad (2.37)$$

The proof is similar to that of 2.16. The only difference is that in a) and b) the estimate (2.17) may be replaced by the sharper bound (2.19).

**Lemma 2.18.** *The notation of 2.15 and 2.16 is understood. Let  $\alpha_0, \dots, \alpha_{2n+1} \in q(TA)$  be elements of the form*

$$\alpha_i = p(\varrho \omega^{n_0}) q(\varrho \omega^{n_1}) \dots q(\varrho \omega^{n_{k_i}}) (a^0, \dots, a^{k_i+2l_i}),$$

$$a^0, \dots, a^{k_i+2l_i} \in S, \quad l_i = (n_0 + \dots + n_{k_i}) \geq 0, \quad k_i > 0,$$

and put  $k = k_0 + \dots + k_{2n+1}$ ,  $l = l_0 + \dots + l_{2n+1}$ . Let  $P$  be a spectral projection associated to the self adjoint operator  $\mathcal{D}^2$  and put  $\eta = \text{Trace}(P e^{-\frac{1}{4}\mathcal{D}^2})$ . Then the odd part of the operator valued distribution

$$T^{\mu_{2n+1}} = T_+^{\mu_{2n+1}} + \lambda^{\frac{1}{2}} T_-^{\mu_{2n+1}} = P \Phi^P(\alpha_0) P \dots P \Phi^P(\alpha_{2n+1}) P \quad (2.38)$$

can be written as a sum

$$T_-^{\mu_{2n+1}} = \sum_{j=0}^{n+1} \lambda^j T_j^{\mu_{2n+1}} \quad (2.39)$$

of distributions which satisfy

$$T_j^{\mu_{2n+1}} \underset{\frac{1}{128}}{\prec} (C_{32}(\mathcal{E}) \parallel S \parallel)^{k+2l+2n+2} f_{l+1}(s) \quad (2.40)$$

Suppose in addition that  $\text{Im}(P) \cap \text{Ker}(\mathcal{D}) = 0$  and that  $\eta < e^{-\frac{1}{4}}$ . Then

$$T_j^{\mu_{2n+1}} \underset{\frac{1}{128}}{\prec} (C_{33}(\mathcal{E}) \parallel S \parallel)^{k+2l+2n+2} (-4 \log \eta)^{-j} \eta^s f_{l+1}(s) \quad (2.41)$$

This follows in a tedious but straightforward calculation from 2.9, 2.16 and 2.17. It should be noted that the estimates in (2.40) are sufficient for our purpose but far from being sharp. Lemma 2.16 and 2.17 are much more precise but we prefer to give a very simple final formula in (2.40).

### Free products

Before we proceed we recall a few facts about locally convex topologies on the various universal algebras we are going to use.

It is easy to see that colimits (like free products for example) do not necessarily exist in the category of Banach algebras. This obstacle disappears if one passes to a smaller category, the category of Banach algebras and contractive homomorphisms (i.e. homomorphisms of norm less or equal too one) as morphisms. We do not discuss this in full generality but treat only the cases we are interested in.

**Definition 2.19.** Let  $A$  be a Banach algebra let  $R \geq 1$  be a real number. We denote by  $Q_R A$  the free product (in the category of Banach algebras and contractive homomorphisms) of two copies of  $A$  equipped with the rescaled norm  $R \parallel - \parallel_A$ .

The Banach algebra  $Q_R A$  is thus the completion of the algebraic free product  $QA = A * A$  with respect to the largest submultiplicative seminorm for which the canonical inclusions  $\theta : A \rightarrow QA$ ,  $\theta^\gamma : A \rightarrow QA$  satisfy

$$\parallel \theta \parallel \leq R, \quad \parallel \theta^\gamma \parallel \leq R$$

By construction  $Q_R A$  possesses the following universal property: there is a canonical bijection between the set of pairs of homomorphisms of norm less or equal to  $R$  from  $A$  to some Banach algebra  $B$  and the set of contractive homomorphisms (i.e. homomorphisms of norm less or equal to 1) from  $Q_R A$  to  $B$ . There are canonical contractive homomorphisms  $Q_{R'} A \rightarrow Q_R A$  for  $R' > R$  and the inverse limit

$$Q_{top} A = \lim_{\infty \leftarrow R} Q_R A \quad (2.42)$$

is a Fréchet algebra. It contains the algebraic free product  $QA$  as dense subalgebra and possesses an obvious universal property.

There exists an extension

$$0 \longrightarrow q_RA \longrightarrow Q_RA \xrightarrow{id*id} A \longrightarrow 0 \quad (2.43)$$

with canonical linear multiplicative sections  $\theta$  and  $\theta^\gamma$  of norm less or equal to  $R$ . It is natural in an appropriate sense. Similarly there is a splitting extension

$$0 \longrightarrow q_{top}A \longrightarrow Q_{top}A \xrightarrow{id*id} A \longrightarrow 0$$

of Fréchet algebras. We denote by

$$\iota_A^\alpha \in HC_0^\alpha(A, q_RA), \quad R \geq 1, \quad \alpha = \epsilon, anal, loc \quad (2.44)$$

the natural element corresponding to the class

$$\theta_* - \theta_*^\gamma \in Ker(HC_0^\alpha(A, Q_RA) \rightarrow HC_0^\alpha(A, A)) \quad (2.45)$$

under the excision isomorphism in entire, analytic, or local cyclic cohomology, respectively.

We pass now to locally convex topologies on tensor algebras.

**Definition 2.20.** [Pu2]. Let  $A$  be a Banach algebra and let  $R \geq 1$  be a real number. The Banach algebra  $T_RA$  is the completion of the tensor algebra  $TA$  over  $A$  with respect to the largest submultiplicative seminorm for which the canonical linear inclusion  $\varrho : A \rightarrow TA$  and the bilinear curvature map  $\omega : A \otimes_\pi A \rightarrow TA$  satisfy

$$\|\varrho\| \leq 2, \quad \|\omega\| \leq \frac{1}{R}$$

The completed tensor algebra  $T_RA$  is natural in  $A$  with respect to contractive algebra homomorphisms. The identity on  $TA$  extends to a contractive homomorphism  $T_RA \rightarrow T_{R'}A$  for  $R < R'$ . The formal inductive limit

$$\mathcal{T}A = \lim_{R \rightarrow \infty} T_RA \quad (2.46)$$

is called the "strict" universal infinitesimal deformation of  $A$ . Its properties are described in detail in [Pu2], section 1.

Now we can formulate

**Proposition 2.21.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . Let  $\Phi : Q(TA) \rightarrow \tilde{\mathcal{L}}$  be the characteristic morphism (2.10) associated to  $\mathcal{E}$  and let  $\tilde{\tau}$  be the odd trace on  $\tilde{\mathcal{L}}$  introduced in 2.3. Then the pull-back of  $\tilde{\tau}$  along  $\Phi$  extends to a continuous trace*

$$\Phi^*\tilde{\tau} : Q_R(\mathcal{T}A) \rightarrow \mathbb{C}$$

*on the ind-Banach algebra  $Q_R(\mathcal{T}A) = \lim_{R' \rightarrow \infty} Q_R(T_{R'}A)$  provided that  $R \gg 0$  is large enough.*

**Proof:** Unfortunately one cannot make use of the characteristic properties of the involved universal algebras because for us  $\tilde{\mathcal{L}}$  is just an abstract algebra without any distinguished topology. Therefore the continuity of  $\Phi$  does not make any sense and one can only talk about the continuity of  $\Phi^*\tilde{\tau}$ . To this end we want to make the norms on  $Q_R(T_{R'}A)$  as explicit as possible. We begin with a general remark. If  $A$  is a complex algebra and if  $S \subset A$  is a subset generating  $A$  as an algebra, then there exists a largest submultiplicative seminorm on  $A$  satisfying  $\|S\| \leq 1$ . In fact this seminorm is given by

$$\|a\| = \inf_{a = \sum \lambda_i s_i} \sum |\lambda_i| \quad (2.47)$$

where the infimum is taken over all presentations  $a = \sum_{finite} \lambda_i s_i$  with  $\lambda_i \in \mathbb{C}$  and  $s_i \in S^\infty$ , the multiplicative closure of  $S$ .

Let now  $A$  be a Banach algebra and fix  $R \geq 1$ . For  $m \in \mathbb{N}$  let  $\|\cdot\|_{(R,m)}$  be the largest seminorm on  $QA = A * A$  satisfying

$$\|pa^0qa^1 \dots qa^n\|_m \leq (2R^2 + n)^m R^n \|a^0\|_A \dots \|a^n\|_A \quad (2.48)$$

These norms are not submultiplicative but satisfy

$$\|\alpha\alpha'\|_{(R,m)} \leq \|\alpha\|_{(R,m+1)} \|\alpha'\|_{(R,m)} \quad (2.49)$$

for all  $\alpha, \alpha' \in QA$ . A straightforward calculation shows that these seminorms are related to the submultiplicative norms of 2.19 by

$$\|\cdot\|_{(R,0)} \leq \|\cdot\|_{QA_{R'}} \quad (2.50)$$

provided that  $R' \geq 3R^3$ . In fact consider (2.47) in the case of the algebra  $QA$  and the generating subset  $S = \frac{1}{R'}(\theta(U) \cup \theta^\gamma(U))$  where  $U$  denotes the unit ball of  $A$ . Then (2.47) gives an explicit formula for the submultiplicative norm on  $QA_{R'}$ . Now for any  $s = \prod s_i \in S^\infty$  one finds  $\|s\|_{(R,0)} \leq \prod \|s_i\|_{(R,1)} \leq 1$  because  $\|S\|_{(R,1)} \leq 1$ . From this the inequality  $\|\cdot\|_{(R,0)} \leq \|\cdot\|_{R'}$  follows.

In a similar spirit consider for  $R \geq 1$  and  $m \in \mathbb{N}$  the tensor algebra  $TA$  equipped with the largest seminorm on  $TA$   $\|\cdot\|_{(R,m)}$  [Pu], 5.6, [Pu2], 1.22 which satisfies

$$\|\varrho(a^0)\omega(a^1, a^2) \dots \omega(a^{2n-1}, a^{2n})\|_{(R,m)} \leq (2+2n)^m R^{-n} \|a^0\|_A \dots \|a^{2n}\|_A \quad (2.51)$$

Again these norms are not submultiplicative but satisfy [Pu2], 1.22

$$\|\alpha\alpha'\|_{(R,m)} \leq \|\alpha\|_{(R,m+1)} \|\alpha'\|_{(R,m)} \quad (2.52)$$

One deduces as before that these seminorms on  $TA$  are related to the submultiplicative norms of (2.47) by

$$\|\cdot\|_{(R,0)} \leq \|\cdot\|_{TA_{R'}} \quad (2.53)$$

provided that  $R \geq 4R'$ . The present proposition claims that for sufficiently large but fixed  $R > 1$  and all  $R_1 > 1$  the functional  $\Phi^*\tilde{\tau}$  is a continuous trace on  $Q_R(T_{R_1}A)$ .

According to (2.50) this would follow from the estimates  $|\Phi^*\tilde{\tau}(x)| \leq C(R_1) \|x\|_{(R,0)}$ . By definition of this seminorm it suffices to verify the estimates

$$|\Phi^*\tilde{\tau}(pq^k(\alpha_0, \dots, \alpha_n))| \leq C(R_1) R^k \prod \|\alpha_i\|_{T_{R_1}A} \quad (2.54)$$

By (2.53) the bound (2.54) would follow from a similar estimate with respect to the seminorms (2.51) instead of  $\|\cdot\|_{T_{R_1}A}$  so that the claimed continuity of  $\Phi^*\tilde{\tau}$  would be finally a consequence of the estimates

$$|\Phi^*\tilde{\tau}(p(\varrho\omega^{n_0})q(\varrho\omega^{n_1})\dots q(\varrho\omega^{n_k}))| \leq C(R_2) R^k \prod \|\varrho\omega^{n_i}\|_{(R_2,0)} \quad (2.55)$$

for  $\varrho\omega^{n_0}, \dots, \varrho\omega^{n_k} \in TA$  where  $\|\varrho\omega^n(a^0, \dots, a^{2n})\|_{(R_2,0)} = R_2^{-n} \prod \|a^i\|_A$ .

Recall that if  $T = T_+ + \lambda^{\frac{1}{2}}T_-$ ,  $T_{\pm} \in \mathcal{L}$  is an operator valued distribution in  $\tilde{\mathcal{L}}$ , whose odd part possesses a majorizing function  $T_- \prec_r f$ , then by definition

$$|\tilde{\tau}(\lambda^j T)| \leq C(r)^j j! f(1) \quad (2.56)$$

for all  $j \geq 0$ . We will make use of (2.25) and (2.26) to obtain the desired estimates. So for

$$\begin{aligned} \alpha &= p(\varrho\omega^{n_0})q(\varrho\omega^{n_1})\dots q(\varrho\omega^{n_k})(a^0, \dots, a^{k+2l}), \\ a^0, \dots, a^{k+2l} &\in S \subset A, l = (n_0 + \dots + n_k) > 0 \end{aligned}$$

and any  $R \geq 1$  we obtain with  $l = 2r + 1$  or  $l = 2r + 2$

$$\begin{aligned} |\tilde{\tau}(\Phi(\alpha))| &\leq (C_{34}(\mathcal{E}) \|S\|)^{k+2l+1} f_l(1) \\ &\leq (C_{35} \|S\|)^{k+4r+1} \frac{1}{r!} \leq R_2^{-2r} (C_{35} \|S\|)^{k+1} \frac{(R_2^2 (C_{35} \|S\|)^4)^r}{r!} \\ &\leq C_{36}(\mathcal{E}, R_2, \|S\|) R_2^{-(2r+2)} (C_{35}(\mathcal{E}) \|S\|)^{k+1} \end{aligned}$$

If we take for  $S$  the unit ball in  $A$  and have a look at (2.51) and (2.55) we see that we just obtained the required estimate provided that  $R > C_{35}(\mathcal{E})$ . Consider now elements  $\alpha \in Q(TA)$  of the form

$$\alpha' = p(\varrho)q(\varrho)^k(a^0, \dots, a^k), a^0, \dots, a^k \in S, k > 0$$

Then according to (2.29)  $|\Phi(\alpha')| \leq (C_{37} \|S\|)^{k+1} f_1(s)$  which implies  $|\Phi(\alpha')| \leq C_{38}(\mathcal{E}) \|\alpha'\|_{(R,0)}$  for  $R > C_{37}$ . The case of the distributions  $\alpha'' = q(\varrho)^n$  is treated in the same way and the last case  $\alpha = p(\varrho(a))$  follows easily from (2.29).  $\square$

We are able to give a first alternative description of the cohomology class of Connes' character.

**Theorem 2.22.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $Ch_{\epsilon}(\mathcal{E}) \in CC_{\epsilon}^0(A)$  be its character in the sense of Connes (see 1.5). Let  $\iota_{TA}^{\epsilon} \in HC_0^{\epsilon}(TA, q_R(TA))$ ,  $R \gg 0$  be the canonical bivariant cohomology class (2.44) associated by excision to the universal splitting extension. Let finally  $\Phi^*(\tilde{\tau}) = \Phi_* \circ \tilde{\tau}$  be the trace on  $Q_R(TA)$  constructed in 2.21 which is well defined for  $R \gg 0$  sufficiently large. Then*

$$\pi_* \circ [Ch_{\epsilon}(\mathcal{E})] = \iota_{TA}^{\epsilon} \circ [\Phi_* \circ \tilde{\tau}] \in HC_0^{\epsilon}(TA, \mathbb{C}) = HC_{\epsilon}^0(TA) \quad (2.57)$$

**Proof:** Because  $\Phi_* \circ \tilde{\tau} \in HC_0^\epsilon(q(\mathcal{T}A), \mathbb{C})$  is the restriction of a trace on  $Q(\mathcal{T}A)$  one has  $\iota_{\mathcal{T}A}^\epsilon \circ \Phi_* \circ \tilde{\tau} = (\theta_* - \theta_*^\gamma) \circ \Phi_* \circ \tilde{\tau} = \theta_* \circ \Phi_* \circ \tilde{\tau} - \theta_*^\gamma \circ \Phi_* \circ \tilde{\tau}$ . By proposition 2.21 and Connes' theorem 1.5  $\pi_* \circ Ch_\epsilon(\mathcal{E})$  and  $\theta_* \circ \Phi_* \circ \tilde{\tau} - \theta_*^\gamma \circ \Phi_* \circ \tilde{\tau}$  are entire cyclic cocycles on  $\mathcal{T}A$ . It suffices therefore to show that they agree on the dense subcomplex  $CC_*(\mathcal{T}A)$  of  $CC_*^\epsilon(\mathcal{T}A)$ . This is true by 1.12.  $\square$

### Spectra of $\theta$ -summable modules

The  $\theta$ -summability condition allows Connes to use a heat kernel regularization to obtain his character formula. We want to replace the heat kernel regularization by a cutoff regularization which uses spectral projections of the operator  $\mathcal{D}^2$ . Such regularizations occur in the (bivariant) character formulas of [Pu], [Me]. They apply to arbitrary Fredholm modules but do not lead to natural explicit formulas. In order to carry out this change of regularization some information about the spectrum of the operator  $\mathcal{D}^2$  is needed.

**Lemma 2.23.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module. Put*

$$N_k := \#\{\lambda \in Sp(\mathcal{D}^2), k \leq \lambda < k+1\} \quad (2.58)$$

*Then for every  $\alpha > 0$  there exists a constant  $C_\alpha$  such that*

$$N_k \leq C_\alpha (1 + \alpha)^k, \quad \forall k \geq 0 \quad (2.59)$$

**Proof:** By assumption the resolvent  $(1 + \mathcal{D}^2)^{-1}$  is a compact selfadjoint operator. This implies that the spectrum of  $\mathcal{D}^2$  is discrete with finite multiplicities. Let  $0 \leq \lambda_0 \leq \lambda_1 \leq \dots$  be the eigenvalues of  $\mathcal{D}^2$  counted with multiplicities. Let  $t_0 > 0$  be such that  $e^{t_0} < 1 + \alpha$ . Then

$$\sum_{k=0}^{\infty} N_k (1 + \alpha)^{-k} \leq \sum_{k=0}^{\infty} N_k e^{-t_0 k} \leq \text{Trace}(e^{-t_0 \mathcal{D}^2}) < \infty$$

which implies the claim.  $\square$

**Lemma 2.24.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module. There exists a sequence  $P_0 \leq P_1 \leq \dots \leq P_n \leq \dots$  of finite rank projections in  $\mathcal{L}(\mathcal{H})$  which possesses the following properties.*

$$\lim_{n \rightarrow \infty} \text{Rank}(P_n) = \infty, \quad \overline{\lim}_{n \rightarrow \infty} (\text{Rank}(P_n))^{\frac{1}{n}} < \infty$$

$$[\mathcal{D}, P_n] = 0, \quad [\epsilon, P_n] = 0, \quad \forall n \in \mathbb{N}$$

$$\lim_{n \rightarrow \infty} \left( \| (1 - P_n) e^{-\frac{\mathcal{D}^2}{4}} \|_1 \right)^{\frac{1}{n}} = 0$$

**Proof:** For  $k \in \mathbb{N}$ ,  $k > 2$  consider the sequence of spectral projections  $P_{k,n} = P_{[0, kn]}(\mathcal{D}^2)$ . Choose  $\alpha_k > 0$  such that  $(1 + \alpha_k)^k \leq 2$ . By lemma 2.23 we find

$$\text{Rank}(P_{k,n}) = \sum_{j=0}^{kn-1} N_j \leq C_{\alpha_k} \sum_{j=0}^{kn-1} (1 + \alpha_k)^j \leq \alpha_k^{-1} C_{\alpha_k} 2^n$$

and

$$\text{Trace}((1 - P_{k,n}) e^{-\frac{p^2}{4}}) \leq \sum_{j=kn}^{\infty} N_j e^{-\frac{j}{4}} \leq C'_{\alpha_k} (((1 + \alpha_k) e^{-\frac{1}{4}})^k)^n$$

Choosing an appropriate subsequence of  $(P_{k,n})_{k,n \in \mathbb{N}}$  yields a sequence with the desired properties.  $\square$

We are now ready to replace the heat kernel regularization, which is central in Connes' construction of the character of  $\theta$ -summable Fredholm modules by a more primitive cut-off regularization. This has two advantages: it applies to Fredholm modules without any summability condition and provides much more flexibility with respect to algebraic operations, homotopies, and so on. Because we want to identify Connes' character not only with the universal bivariant Chern-Connes character in local cyclic cohomology but also with Meyer's Chern character in analytic cyclic cohomology, we use a change of regularization which identifies the Connes character formula directly with Meyer's character and show finally that it coincides with our character in the local cyclic theory.

**Lemma 2.25.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . Choose a sequence  $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$  of finite rank projections in  $\mathcal{L}(\mathcal{H})$  which satisfies the conditions of 2.24. Define an odd cochain  $\Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}}) = \Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1})_{n \in \mathbb{N}}$  on the (acyclic)  $(b, B)$ -bicomplex  $CC(q(TA))$  by*

$$\Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1}(\alpha_0, \dots, \alpha_{2n+1}) = \tilde{\mu}_{\mathcal{P}}^{2n+1}(\Phi_{\mathcal{P}}(\alpha^0), \dots, \Phi_{\mathcal{P}}(\alpha^{2n+1})) \quad (2.60)$$

where  $\tilde{\mu}_{\mathcal{P}} \in CC^1(\tilde{\mathcal{L}})$  is the cochain of 2.4 and the linear map  $\Phi_{\mathcal{P}} : q(TA) \rightarrow \tilde{\mathcal{L}}$  was introduced in 2.17. Then for sufficiently large  $R \gg 0$  the functionals  $\Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}}) = \Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1})_{n \in \mathbb{N}}$  define an entire cyclic cochain on the ind-Banach algebra  $q_R(\mathcal{T}A) = \varinjlim_{R' \rightarrow \infty} q_R(T_{R'}A)$ .

**Proof:** A reasoning similar to the one in the proof of 2.21 and the definition [Co] of entire cyclic cohomology show that it suffices to verify the following. For some  $R_0 \geq 1$  and all  $R \geq 1$  there exist constants  $C_{2n+1}(\mathcal{E}, R) > 0$ ,  $n > 0$ , such that for all bounded subsets  $S \subset A$

$$|\Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1}(\alpha_0, \dots, \alpha_{2n+1})| \leq C_{2n+1}(\mathcal{E}, R) R_0^k R^{-l} \|S\|^{2n+2+2l+k} \quad (2.61)$$

for all elements  $\alpha_0, \dots, \alpha_{2n+1} \in q(TA)$  of the form

$$\begin{aligned} \alpha_i &= p(\varrho \omega^{n_0}) q(\varrho \omega^{n_1}) \dots q(\varrho \omega^{n_{k_i}}) (a^0, \dots, a^{k_i+2l_i}) \\ a^0, \dots, a^{k_i+2l_i} &\in S, l_i = (n_0 + \dots + n_{k_i}) \geq 0, k_i > 0, \end{aligned}$$

$k = k_0 + \dots + k_{2n+1}$ ,  $l = l_0 + \dots + l_{2n+1}$ . Moreover these constants have to satisfy the asymptotic decay condition

$$\overline{\lim}_{n \rightarrow \infty} (n! C_{2n+1}(\mathcal{E}, R))^{\frac{1}{n}} = 0 \quad (2.62)$$

for all  $R \geq 1$ . The estimates of lemma 2.18 together with 2.11 show that

$$\begin{aligned} n! |\Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1}(\alpha_0, \dots, \alpha_{2n+1})| &\leq \sum_{j=0}^{n+1} C_{37}^j j! (C_{32}(\mathcal{E}) \|S\|)^{k+2l+2n+2} f_{l+1}(1) \\ &\leq C_{38}^n n! C_{32}^k R^{-l} \exp(C_{39} R^2) \|S\|^{k+2l+2n+2} \end{aligned}$$

which gives an estimate of the type (2.61). It is however not strong enough to guarantee the decay condition (2.62). To achieve this we suppose that  $n \gg 0$  is large enough that  $Im(1 - P_n) \cap Ker \mathcal{D} = 0$  and  $\eta_n = Trace((1 - P_n) e^{-\frac{\mathcal{D}^2}{4}}) < e^{-\frac{1}{4}}$ . Then we can use the sharper estimate of (2.41) and arrive at

$$\begin{aligned} &n! |\Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1}(\alpha_0, \dots, \alpha_{2n+1})| \\ &\leq \left( \sum_{j=0}^{n+1} C_{37}^j j! (-4 \log \eta_n)^{-j} \right) (C_{33}(\mathcal{E}) \|S\|)^{k+2l+2n+2} \eta_n f_{l+1}(1) \end{aligned} \quad (2.63)$$

According to lemma 2.24  $\lim_{n \rightarrow \infty} (\eta_n)^{\frac{1}{n}} = 0$  so that  $\lim_{n \rightarrow \infty} \frac{-\log \eta_n}{n} = \infty$ . Thus for  $n \gg 0$  large enough  $(-4 \log \eta_n) \geq 2C_{37}(n+1)$  and then

$$\sum_{j=0}^{n+1} C_{37}^j j! (-4 \log \eta_n)^{-j} \leq \sum_{j=0}^{n+1} 2^{-j} \frac{j!}{(n+1)^j} \leq 2 \quad (2.64)$$

For the other term in (2.63) we arrive at

$$(C_{33}(\mathcal{E}) \|S\|)^{k+2l+2n+2} \eta_n f_{l+1}(1) \leq (\exp(C_{40} R^2) \eta_n) R^{-l} C_{33}^{2n+2+k} \|S\|^{k+2l+2n+2}$$

which in view of 2.24 yields the desired bound provided that  $R_0 \geq C_{33}(\mathcal{E})$ .  $\square$

### Pointwise estimates for the operator norm of operator valued holomorphic functions

We take up again the study of continuity properties of various cyclic cochains. We will not only use uniform estimates of the Schatten norms of the involved distributions near 0 but also uniform pointwise estimates of the operator norm of their Laplace transforms.

The contour  $\Gamma$  of (2.4) divides the plane into two parts. Following Connes [Co] we are interested in pointwise norm estimates of various operator valued holomorphic functions on the closure of the connected component of  $\mathbb{C} \setminus \Gamma$  contained in the right halfplane. Explicitely it is given by

$$\overline{U} = \{z \in \mathbb{C}, |Im(z)| \leq \sqrt{3} Re(z-1)\} \quad (2.65)$$

**Lemma 2.26.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $F(z) = (\mathcal{D} + z^{\frac{1}{2}} \epsilon)(\mathcal{D}^2 + z)^{-\frac{1}{2}}$  be the Laplace transform of the associated involutive distribution. Then*

$$\|F(z)\|_{\mathcal{L}(\mathcal{H})} \leq C_{41}, \quad \| [F(z), a] \|_{\mathcal{L}(\mathcal{H})} \leq C_{42}(\mathcal{E}) |z|^{-\frac{1}{2}} \|a\|_A$$



for all  $a \in A$  and  $z \in \overline{U}^2$ . Similar estimates hold for the Laplace transforms of the even respectively odd parts (with respect to the grading of  $\tilde{\mathcal{L}}$ ) of the involved distributions.

**Proof:** The decomposition of the operator under consideration into even and odd parts

$$F(z) = (\mathcal{D} + \epsilon z^{\frac{1}{2}})(\mathcal{D}^2 + z)^{-\frac{1}{2}} = F_1(z^{-\frac{1}{2}}\mathcal{D}) + \epsilon F_2(z^{-\frac{1}{2}}\mathcal{D}) \quad (2.66)$$

is given by operators obtained from  $\mathcal{D}' = z^{-\frac{1}{2}}\mathcal{D}$  by holomorphic functional calculus with respect to

$$F_1(u) = u(1 + u^2)^{-\frac{1}{2}}, \quad F_2(u) = (1 + u^2)^{-\frac{1}{2}} \quad (2.67)$$

By assumption the operator  $\mathcal{D}$  is selfadjoint and its resolvent  $(1 + \mathcal{D}^2)^{-1} \in \mathcal{K}(\mathcal{H})$  is compact. Therefore the spectrum of  $\mathcal{D}$  is real and discrete with finite multiplicities. One deduces easily that there are universal bounds

$$\|F_1(z^{-\frac{1}{2}}\mathcal{D})\|_{\mathcal{L}(\mathcal{H})} \leq C_{43}, \quad \|F_2(z^{-\frac{1}{2}}\mathcal{D})\|_{\mathcal{L}(\mathcal{H})} \leq C_{44} \quad (2.68)$$

for  $z \in \overline{U}^2$  from which the first assertion follows. Our analysis of the commutators  $[F(z), a]$ ,  $a \in A$ , is based on the identity [BJ]

$$(1 + \mathcal{D}^2)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} (1 + \mathcal{D}^2 + \lambda)^{-1} d\lambda \quad (2.69)$$

For the commutator of the even part of  $F$  and  $a \in A$  we find

$$\begin{aligned} [F(z)_+, a] &= [F_1(z^{-\frac{1}{2}}\mathcal{D}), a] = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} [\mathcal{D}', a] (1 + \mathcal{D}'^2 + \lambda)^{-1} d\lambda \\ &\quad - \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} (\mathcal{D}' (1 + \mathcal{D}'^2 + \lambda)^{-1} [\mathcal{D}', a] \mathcal{D}' (1 + \mathcal{D}'^2 + \lambda)^{-1} d\lambda \\ &\quad - \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \mathcal{D}'^2 (1 + \mathcal{D}'^2 + \lambda)^{-1} [\mathcal{D}', a] (1 + \mathcal{D}'^2 + \lambda)^{-1} d\lambda \end{aligned}$$

To estimate the norms of these operators we observe that for all  $u \in \overline{U}$  and all  $\lambda \in \mathbb{R}_+$  one has  $|(1 + u^2 + \lambda)^{-1}| \leq 2(1 + \lambda)^{-1}$ ,  $|u(1 + u^2 + \lambda)^{-1}| \leq 2(1 + \lambda)^{-\frac{1}{2}}$  and  $|u^2(1 + u^2 + \lambda)^{-1}| \leq 2$ . With the help of these bounds we deduce

$$\|[F(z)_+, a]\| \leq C_{45} \|\mathcal{D}', a\| = C_{46} |z|^{-\frac{1}{2}} \|\mathcal{D}, a\| \quad (2.70)$$

for all  $z \in \overline{U}^2$ . The corresponding assertion for the odd part of  $F(z)$  follows from similar arguments.  $\square$

**Lemma 2.27.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $\Phi : q(TA) \rightarrow \tilde{\mathcal{L}}$  be its characteristic homomorphism introduced in 2.6. Let  $P$  be a spectral projection associated to  $\mathcal{D}^2$  and suppose that  $\text{Im}(P) \cap$*

$\ker(\mathcal{D}) = 0$  and that  $\eta = \text{Trace}(P e^{-\frac{\mathcal{D}^2}{4}}) < 1$ . Then the image of the distribution  $\Phi(q(\varrho(a))P, a \in A$ , under the Laplace transform, as well as its even and odd parts satisfy

$$\| \Phi(q(\varrho(a))(z)P \| \leq C_{47} \left( (-\log \eta)^{-\frac{1}{2}} |z|^{\frac{1}{2}} + |z|^{-\frac{1}{2}} \right) \| a \| \quad (2.71)$$

for all  $a \in A, z \in \overline{U}^2$ .

**Proof:** We calculated in (2.30) the distribution  $\Phi(q(\varrho(a)))$ . In fact

$$\Phi(q(\varrho(a))(z)P = \left( \frac{1}{2} a F(z)_- + \frac{1}{4} [F(z)_-, a] - \frac{1}{4} F(z)_- [\mathcal{D}, a] (\mathcal{D}^2 + z)^{-\frac{1}{2}} \right) P$$

According to 2.26  $\| \frac{1}{4} [F(z)_-, a] - \frac{1}{4} F(z)_- [\mathcal{D}, a] (\mathcal{D}^2 + z)^{-\frac{1}{2}} \| \leq C_{48} |z|^{-\frac{1}{2}} \| a \|$ . On the other hand  $\| F(z)_- P \| = \| z^{\frac{1}{2}} \epsilon (\mathcal{D}^2 + z)^{-\frac{1}{2}} P \| \leq 2 |z|^{\frac{1}{2}} \lambda_1^{-\frac{1}{2}}$  where  $\lambda_1$  is the smallest nonzero eigenvalue of  $\mathcal{D}^2 P$ . It satisfies  $\lambda_1 \geq -4 \log \eta$  (see the proof of 2.12) whence the result.  $\square$

The estimates obtained so far yield

**Proposition 2.28.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $\Phi : q(TA) \rightarrow \tilde{\mathcal{L}}$  be its characteristic homomorphism (see (2.10)). Choose a sequence of finite rank projections in  $\mathcal{L}(\mathcal{H})$  which satisfies the conditions of 2.24 and consider the odd cochain  $\tilde{\mu}_{\mathcal{P}} = (\mu^{2n+1})_{n \in \mathbb{N}} \in CC^1$  (2.6), on the  $(b, B)$ -bicomplex of  $\tilde{\mathcal{L}}$ . Then the pull-back of this functional along  $\Phi$  extends to an entire cyclic cochain*

$$\Phi^*(\tilde{\mu}_{\mathcal{P}}) \in CC_{\epsilon}^1(q_{R_0}(TA))$$

on the ind-Banach algebra  $q_{R_0}(TA) = \lim_{R \rightarrow \infty} q_{R_0}(T_R A)$  provided that  $R_0 \gg 0$  is large enough.

**Proof:** It will not be shown directly that  $\Phi^*(\tilde{\mu}_{\mathcal{P}})$  is an entire cyclic cochain but we will rather prove that the difference  $\Phi^*(\tilde{\mu}_{\mathcal{P}}) - \Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})$  (see 2.17) is entire. The conclusion follows then in conjunction with 2.25. As we have seen in the proofs of 2.21 and 2.25 it suffices to verify the following. There exists some  $R_0 \geq 1$  such that for all  $R_1, R_2 \geq 1$  there are constants  $C(\mathcal{E}, R_1, R_2) > 0$  so that for every bounded subset  $S \subset A$

$$n! |(\Phi^*(\tilde{\mu}_{\mathcal{P}})^{2n+1} - \Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1})(\alpha_0, \dots, \alpha_{2n+1})| \leq C(R_1, R_2) R_0^k R_1^{-l} R_2^{-n} \| S \|^{2n+2+2l+k} \quad (2.72)$$

for all  $n \geq 0$  and all elements  $\alpha_0, \dots, \alpha_{2n+1} \in q(TA)$  of the form

$$\alpha_i = p(\varrho \omega^{n_0}) q(\varrho \omega^{n_1}) \dots q(\varrho \omega^{n_{k_i}}) (a^0, \dots, a^{k_i+2l_i})$$

$$a^0, \dots, a^{k_i+2l_i} \in S, l_i = (n_0 + \dots + n_{k_i}) \geq 0, k_i > 0,$$

$$k = k_0 + \dots + k_{2n+1}, l = l_0 + \dots + l_{2n+1}.$$

To understand the algebraic manipulations which follow note that for any idempotent  $P \in A$  and any elements  $a^0, \dots, a^n \in A$  one has the identity

$$P\left(\prod_0^n a^i\right)P - \prod_0^n (Pa^iP) = \sum_{j=0}^{n-1} P\left(\prod_0^j a_i\right)[a^{j+1}, P]\left(\prod_{j+2}^n Pa_kP\right)P \quad (2.73)$$

In the sequel the notations of lemma 2.17 are understood. For  $\alpha = p(\varrho\omega^{n_0})q(\varrho\omega^{n_1})\dots q(\varrho\omega^{n_k})(a^0, \dots, a^{k+2l})$ ,  $a^0, \dots, a^{k+2l} \in S$ ,  $l_i = (n_0 + \dots + n_k) \geq 0$ ,  $k > 0$  one finds

$$n! \Phi^*(\tilde{\mu}_{\mathcal{P}})^{2n+1}(\alpha_0, \dots, \alpha_{2n+1}) = \tilde{\tau} \left( \prod_{i=0}^{2n+1} (\Phi^{(1-P_n)}(\alpha_i) - \beta_i) \right) \quad (2.74)$$

$$\begin{aligned} n! \Phi_{\mathcal{P}}^*(\tilde{\mu}_{\mathcal{P}})^{2n+1}(\alpha_0, \dots, \alpha_{2n+1}) &= \tilde{\tau} \left( \prod_{i=0}^{2n+1} \Phi^{(1-P_n)}(\alpha_i) \right) \\ \beta &= \Phi^{(1-P_n)}(\alpha) - (1 - P_n) \Phi(\alpha) (1 - P_n) = \\ &= (1 - P_n) \Phi'(p) \Phi^{(1-P_n)}(q^k) (1 - P_n) + \\ &+ (1 - P_n) \sum_{j=1}^k \Phi(pq^{j-1}) \Phi'(q) \Phi^{(1-P_n)}(q^{k-j}) (1 - P_n) \end{aligned} \quad (2.75)$$

$$\Phi'(p) = \frac{1}{2}(T\varphi_0'' + T\varphi_1''), \quad \Phi'(q) = \frac{1}{2}(T\varphi_0'' - T\varphi_1'')$$

$$\begin{aligned} T\varphi_0''(\varrho\omega^m)(a^0, \dots, a^{2m}) &= C_n [P_+ a^0 P_+, Q_n] \prod_1^{2m} (Q_n [F, a^i] Q_n) + \\ &+ C_n \sum_{m'=0}^{2m-1} (P_+ a^0 P_+) \left( \prod_1^{m'-1} [F, a^i] \right) [[F, a^{m'}], Q_n] \left( \prod_{m'+1}^{2m} (Q_n [F, a^i] Q_n) \right) \end{aligned} \quad (2.76)$$

$$\begin{aligned} T\varphi_1''(\varrho\omega^m)(a^0, \dots, a^{2m}) &= C_n \epsilon [P_- a^0 P_-, Q_n] \prod_1^{2m} (Q_n [F, a^i] Q_n) \epsilon + \\ &+ C_n \sum_{m'=0}^{2m-1} \epsilon (P_- a^0 P_-) \left( \prod_1^{m'-1} [F, a^i] \right) [[F, a^{m'}], Q_n] \left( \prod_{m'+1}^{2m} (Q_n [F, a^i] Q_n) \right) \epsilon \end{aligned} \quad (2.77)$$

for  $m > 0$  where  $Q_n = (1 - P_n)$  and  $C_n = \left(-\frac{1}{4}\right)^n$ . For  $m = 0$

$$T\varphi_0''(\varrho)(a^0) = [P_+ a^0 P_+, (1 - P_n)], \quad T\varphi_1''(\varrho)(a^0) = [\epsilon P_- a^0 P_- \epsilon, (1 - P_n)] \quad (2.78)$$

The fact that all commutators with the projection  $(1 - P_n)$  are of finite rank at most  $2 \operatorname{Rank}(P_n)$  enables us to use the Laplace transform and its inverse to control the trace of distributions of the form  $T = T_0[T_1, (1 - P_n)]T_2$ ,  $T_i \in \tilde{\mathcal{L}}$ . Let  $T_i = T_{i+} + T_{i-}$

be the decomposition of  $T_i$  into even and odd parts. Then one finds for any oriented curve  $\Gamma'$  in  $\overline{U}$  (2.65), which is homologous to  $\Gamma = \partial\overline{U}$ , the estimate

$$|\tilde{\tau}(T)| \leq \sum_{\Gamma'} 2 \operatorname{Rank}(P_n) \int_{\Gamma'} \|T_{0\pm}(z^2)\| \|T_{1\pm}(z^2)\| \|T_{2\pm}(z^2)\| |d\mu(z)| \quad (2.79)$$

where the sum runs over all possible choices of even resp. odd parts of the involved operator valued functions with odd total weight. Here  $\| - \|$  denotes the pointwise operator norm of an operator valued holomorphic function.

If we apply this argument in our case with the curve  $\Gamma_\zeta$  given by the oriented boundary of  $U \cap \{z \in \mathbb{C}, \operatorname{Re}(z) > \zeta\}$  we find the following estimate.

$$\begin{aligned} & n! |(\Phi^*(\tilde{\mu}_{\mathcal{P}})^{2n+1} - \Phi_{\mathcal{P}}^*(\tilde{\mu})^{2n+1})(\alpha_0, \dots, \alpha_{2n+1})| \leq \\ & \leq 2 \operatorname{Rank}(P_n) (C_{49}(\mathcal{E}) \|S\|)^{k+2l+2n+2} \sum_{i,j} \zeta^{-(2l+i)} (-\log \eta_n)^{-\frac{j}{2}} \int_{\Gamma_\zeta} |z^2|^{\frac{j}{2}} |d\mu(z)| \end{aligned} \quad (2.80)$$

where the sum runs over the integers  $i, j \geq 0, 0 \leq i+j \leq 2n+2, i+j+l \geq 2n+2$ . Put  $R_0 = C_{49}(\mathcal{E})$ . If one chooses  $\zeta$  such that  $\zeta > C_{49} R_1$  one sees that for each  $n \geq 0$  there exist constants  $C(n, R_1, R_2)$  such that the estimate (2.72) holds. What remains to be done is to show that there is a similar bound which is independent of  $n$ . It suffices to obtain uniform estimates for  $n \gg 0$ . Note that by construction (see 2.24)  $\operatorname{Rank} P_n \leq C_{50}^n$  for a suitable constant  $C_{50}$ . Choose now  $\zeta > R_1 R_2 C_{49}^2 C_{50}$ . As  $\lim_{n \rightarrow \infty} \eta_n^{\frac{1}{n}} = 0$  we may suppose that for  $n \gg 0$  large enough  $-\log \eta_n \geq 4 C_{49}^2 C_{50}^2 R_2 n$ . Then we deduce from (2.80)

$$\begin{aligned} & n! |(\Phi^*(\tilde{\mu}_{\mathcal{P}})^{2n+1} - \Phi_{\mathcal{P}}^*(\tilde{\mu})^{2n+1})(\alpha_0, \dots, \alpha_{2n+1})| \leq \\ & \leq 2 R_0^k R_1^{-l} R_2^{-n} \|S\|^{k+2l+2n+2} \sum_{i,j} \int_{\Gamma_\zeta} \left(\frac{|z^2|}{4n}\right)^{\frac{j}{2}} |d\mu(z)| \end{aligned} \quad (2.81)$$

If one can show that

$$\sup_{n>0} \max_{0 \leq j \leq 2n+2} \int_{\Gamma_\zeta} \left(\frac{|z^2|}{4n}\right)^{\frac{j}{2}} |d\mu(z)| < \infty \quad (2.82)$$

for all  $\zeta \geq 1$  then the desired estimate (2.72) follows.

We may suppose  $\frac{|z^2|}{4n} > 1$ . Then (remember that  $j \leq 2n+2$ ) one has  $(\frac{|z^2|}{4n})^{\frac{j}{2}} \leq (\frac{|z^2|}{4n})^{n+1} \leq \frac{|z^2|^{n+1}}{4^{n+1}(n+1)!} \frac{(n+1)!}{n^{n+1}} \leq 2e^{\frac{|z^2|}{4}}$ . It is easily seen that  $\operatorname{Re}(z^2) \leq -\frac{|z^2|}{3}$  for  $z \in \Gamma_\zeta$  provided that  $|z^2| \gg 0$  is large. Thus finally

$$\sup_{n>0} \max_{0 \leq j \leq 2n+2} \int_{\Gamma_\zeta} \left(\frac{|z^2|}{4n}\right)^{\frac{j}{2}} |d\mu(z)| \leq \int_{\Gamma_\zeta} e^{\frac{|z^2|}{4}} |d\mu(z)|$$

$$\leq \frac{1}{\pi} \int_{\Gamma_\zeta} e^{(\frac{|z|^2}{4} + \operatorname{Re}(z^2))} |dz| \leq \frac{1}{\pi} \int_{\Gamma_\zeta} e^{-\frac{|z|^2}{12}} |dz| + O(1) < \infty$$

which proves the proposition.  $\square$

**Theorem 2.29.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $[Ch_\epsilon(\mathcal{E})] \in HC_\epsilon^0(A)$  be the class of its Connes character 1.5 in entire cyclic cohomology. Choose a sequence  $\mathcal{P}$  of finite rank projections in  $\mathcal{L}(\mathcal{H})$  satisfying the conditions of 2.24 and let  $\tilde{\tau}_{\mathcal{P}} \in CC^0(\tilde{\mathcal{L}})$  be the associated cocycle (2.6). Let  $\Phi : q(TA) \rightarrow \tilde{\mathcal{L}}$  be the characteristic homomorphism (2.10) associated to  $\mathcal{E}$  and denote finally by  $\iota_{TA}^\epsilon \in HC_0^\epsilon(TA, q_R(TA))$ ,  $R \gg 0$  the canonical bivariant cohomology class (2.44) associated to the universal splitting extension. Then for  $R \gg 0$  sufficiently large the functional  $\Phi^*(\tilde{\tau}_{\mathcal{P}})$  extends to an entire cyclic cocycle on the ind-Banach algebra  $q_R(TA)$  and its cohomology class  $[\Phi_* \circ \tilde{\tau}_{\mathcal{P}}] \in HC_0^\epsilon(q_R(TA), \mathbb{C})$  satisfies*

$$\pi_* \circ [Ch_\epsilon(\mathcal{E})] = \iota_{TA}^\epsilon \circ [\Phi_* \circ \tilde{\tau}_{\mathcal{P}}] \in HC_0^\epsilon(TA, \mathbb{C}) = HC_\epsilon^0(TA) \quad (2.83)$$

**Proof:** This is an immediate consequence of 2.22 and 2.28.  $\square$

### 3 The basic conjugation and homotopy

In the previous section we obtained a description of Connes' character of an unbounded  $\theta$ -summable Fredholm module 1.5, which is formally similar to that of the abstract character in  $K$ -homology in (1.27). There is a basic difference however: if one compares the characteristic homomorphisms (1.47) and (1.32) used to define the characters in 2.29 and (1.39), then one observes that the roles of the operators  $F$  and  $\epsilon$  have been exchanged in the corresponding formulas. Modulo compact operators the data defining a bounded Fredholm module are in fact completely symmetric in  $F$  and  $\epsilon$ . A bounded Fredholm module yields a representation of the Clifford algebra of the two-dimensional vector space with orthonormal basis  $\{F, \epsilon\}$  on the Calkin algebra  $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ . There exists an inner automorphism of the Clifford algebra which interchanges the basis elements  $F$  and  $\epsilon$ . It is given by conjugation with the involutive element  $U = \frac{1}{\sqrt{2}}(F + \epsilon)$ . We will study this element in the algebra  $\mathcal{O}$  of holomorphic operator valued functions on  $\mathbb{C} \setminus \mathbb{R}_-$  and will show that conjugation by this element interchanges the characteristic homomorphisms (1.47) and (1.32) modulo a small error. As traces are invariant under conjugation we will obtain a formula for Connes' character which coincides with that of the abstract character modulo a small error. This error can be eliminated by a straightforward homotopy and we arrive at our final description 3.8 of Connes' character class in entire cyclic cohomology.

**Definition 3.1.** Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $F(z)$ ,  $\operatorname{Re}(z) > 0$ , be the associated holomorphic family of

operators (1.5). We define a characteristic family of operators by

$$U(z) = \frac{1}{\sqrt{2}}(F(z) + \epsilon) \quad (3.1)$$

It satisfies

$$U(z) F(z) = \epsilon U(z), \quad U(z) \epsilon = F(z) U(z) \quad (3.2)$$

**Remark 3.2.** *There is an element  $U = \frac{1}{\sqrt{2}}(F + \epsilon) \in \tilde{\mathcal{L}}$  in the associated algebra of operator valued distributions. It is invertible in  $\tilde{\mathcal{L}}$  if and only if  $\text{Ker } \mathcal{D} = 0$ . Therefore conjugation with this element makes no sense among operator valued distributions with controlled growth if  $\mathcal{D}$  possesses a kernel. It turns out however that its Laplace transform  $U(z)$  is invertible among bounded operator valued holomorphic functions (note that the Laplace transform is one to one on  $\mathcal{L}$  but not on  $\tilde{\mathcal{L}}$ ).*

**Lemma 3.3.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module and let  $U(z)$  be the characteristic family of operators (3.1). Then  $U(z)$  is invertible for all  $z \in \mathbb{C} \setminus \mathbb{R}_-$  and*

$$\|U(z)\|_{\mathcal{L}(\mathcal{H})} \leq C_{51}(\mathcal{E}), \quad \|U(z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C_{52}(\mathcal{E}), \quad (3.3)$$

$$\|[U(z), a]\|_{\mathcal{L}(\mathcal{H})} \leq C_{53}(\mathcal{E}) |z|^{-\frac{1}{2}} \|a\|_A \quad (3.4)$$

for all  $z \in \overline{U}^2$  and  $a \in A$ .

**Proof:** Observe that  $[U(z), a] = \frac{1}{\sqrt{2}}[F(z), a]$  and

$$U^2(z) = \frac{1}{2}(2 + (\epsilon F(z) + F(z) \epsilon)) = 1 + z^{\frac{1}{2}}(z + \mathcal{D}^2)^{-\frac{1}{2}} = 1 + F_2(z^{-\frac{1}{2}} \mathcal{D})$$

in the notations of 2.26. The proof is then similar to the one of 2.26.  $\square$

**Definition 3.4.** Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $F(z)$  and  $U(z)$  be the associated families of operators (1.5). Let  $\mathcal{O}$  be the algebra of holomorphic operator valued functions on  $\mathbb{C} \setminus \mathbb{R}_-$  and define families of linear maps  $\rho^t, \rho_{0,1}^t : A \rightarrow \mathcal{O}[t]$  by

$$\rho^t(a) = \rho(a) + t[U(z), \rho(a)]U(z)^{-1} \quad (3.5)$$

$$\rho_0^t(a) = \frac{1+\epsilon}{2} \rho^t(a) \frac{1+\epsilon}{2}, \quad \rho_1^t(a) = F(z) \frac{1-\epsilon}{2} \rho^t(a) \frac{1-\epsilon}{2} F(z) \quad (3.6)$$

The maps  $\rho_0^t$  and  $\rho_1^t$  coincide modulo the ideal  $\mathcal{O}_K \subset \mathcal{O}$  of holomorphic functions with values in the ideal of compact operators.

One has the

**Proposition 3.5.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module and let  $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$  be a family of finite rank projections satisfying the conditions of 2.24. For  $z \in \mathbb{T} \setminus \mathbb{R}_-$  let  $\vartheta_{\mathcal{P}}(z) \in CC^0(\mathcal{O}_{\mathcal{K}})$  be the pullback of the cocycle  $\tau_{\mathcal{P}} \in CC^0(\mathcal{K}(\mathcal{H}))$  (1.37) along the evaluation homomorphism at  $z$ . Let*

$$\Psi_t = T\rho_0^t * T\rho_1^t : q(TA) \rightarrow \mathcal{O}_{\mathcal{K}}[t] \quad (3.7)$$

*be the family of homomorphisms of universal algebras associated to the linear maps of 3.4. Then for fixed  $R \gg 0$  the following holds: for each  $t \in [0, 1]$  the cochain*

$$\nu_t = \int_{\Gamma} \Psi_t^*(\tau_{\mathcal{P}}(z^2)) d\mu(z) \in CC^0(q(TA)) \quad (3.8)$$

*extends to an entire cyclic cocycle*

$$\nu_t \in CC_{\epsilon}^0(q_R(\mathcal{T}A)) \quad (3.9)$$

*The cohomology class of this family of cocycles is independent of  $t$ .*

**Proof:** The map  $\rho^t$  is the sum of the structure homomorphism  $\rho$  and a linear perturbation term  $t[U(z), \rho(a)]U(z)^{-1} = \frac{t}{\sqrt{2}}[F(z), \rho(a)]U(z)^{-1}$  whose pointwise operator-norm is bounded according to 3.3 by  $\| [U(z), \rho(a)]U(z)^{-1} \| \leq C_{53}|z|^{-\frac{1}{2}}$ . As one may replace the path of integration  $\Gamma$  by any of the paths  $\Gamma_{\zeta}$  of (2.80), one may suppose that the perturbation terms  $\rho^t - \rho$  are arbitrarily small in norm. For  $t = 0$  a calculation similar to the one at the end of the proof of 2.28 allows to deduce that  $\nu_0$  is an entire cyclic cocycle. The case  $t \neq 0$  follows by a calculation making use of the case  $t = 0$  and the remark about the norm of the linear perturbation term. The last assertion is a consequence of the Cartan homotopy formula in entire cyclic cohomology.  $\square$

We are going to identify the cohomology class constructed in the previous proposition.

**Lemma 3.6.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . Let  $\Phi^*(\tilde{\tau}_{\mathcal{P}})$  and  $\nu_t, t \in [0, 1]$ , be the entire cyclic cocycles over  $q_R(\mathcal{T}A)$  introduced in 2.29 and 3.5., respectively. Then  $\Phi^*(\tilde{\tau}_{\mathcal{P}}) = \nu_1$  as entire cyclic cocycles.*

**Proof:** It suffices to check that both cocycles coincide on the dense subcomplex  $CC_0(q(TA))$  of  $CC_0^{\epsilon}(q_R(\mathcal{T}A))$ . After applying the Laplace transform one sees that the first cocycle under discussion can be written as

$$\Phi^*(\tilde{\tau}_{\mathcal{P}}) = \int_{\Gamma} \Phi^*(\tau_{\mathcal{P}}(z^2)) d\mu(z)$$

where  $\Phi : q(TA) \rightarrow \mathcal{O}_{\mathcal{K}}$  is the characteristic homomorphism (1.48) of  $\mathcal{E}$ . Denote by  $c(U(z))$  the automorphism of  $\mathcal{O}_{\mathcal{K}}$  given by conjugation with the characteristic

operator valued function  $U(z)$  of (3.1). Because the operator  $U(z)$  commutes with  $\mathcal{D}^2$  it commutes with the projections  $(P_n)_{n \in \mathbb{N}}$  of  $\mathcal{P}$ . The cocycle  $\tau_{\mathcal{P}}$  is thus invariant under conjugation by  $U(z)$ :  $c(U(z))^* \tau_{\mathcal{P}} = \tau_{\mathcal{P}}$ . So we find  $\Phi_* \circ \tau_{\mathcal{P}}(z) = \Phi_* \circ c(U(z))_* \circ \tau_{\mathcal{P}}(z) = (c(U(z)) \circ \Phi)_* \circ \tau_{\mathcal{P}}(z)$ . Now  $c(U(z)) \circ \Phi = (c(U(z)) \circ T\varphi_0) * (c(U(z)) \circ T\varphi_1) = T(c(U(z)) \circ \varphi_0) * T(c(U(z)) \circ \varphi_1)$  because  $c(U(z))$  is a homomorphism of algebras. (The notations are those of (1.47) and (1.32)). Further

$$\begin{aligned} (c(U(z)) \circ \varphi_0)(a) &= U(z) \frac{1+F(z)}{2} \rho(a) \frac{1+F(z)}{2} U(z)^{-1} = \\ &= \frac{1+\epsilon}{2} U(z) \rho(a) U(z)^{-1} \frac{1+\epsilon}{2} = (\rho_0 \circ c(U(z)))(a) \end{aligned}$$

and similarly  $c(U(z)) \circ \varphi_1 = \rho_1 \circ c(U(z))$ . Finally

$$U(z) \rho(a) U(z)^{-1} = \rho(a) + [U(z), \rho(a)] U(z)^{-1} = \rho^1(a)$$

so that

$$c(U(z)) \circ \varphi_0 = \rho_0^1, \quad c(U(z)) \circ \varphi_1 = \rho_1^1$$

in the notations of 3.4. Altogether we have shown that  $c(U(z)) \circ \Phi = \Psi_1 : q(TA) \rightarrow \mathcal{O}_{\mathcal{K}}$ . Therefore

$$\Phi^*(\tau_{\mathcal{P}}(z)) = (c(U(z)) \circ \Phi)^*(\tau_{\mathcal{P}}(z)) = (\Psi_1)^*(\tau_{\mathcal{P}}(z))$$

and the lemma is proved.  $\square$

**Lemma 3.7.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$  and let  $F(z), z \in \mathbb{C} \setminus \mathbb{R}_-$  be the associated holomorphic family of operators. Let*

$$\rho_0, \rho_1 : A \rightarrow \mathcal{O}, \quad \rho_0(a) = \frac{1+\epsilon}{2} \rho(a) \frac{1+\epsilon}{2}, \quad \rho_1(a) = F(z) \frac{1-\epsilon}{2} \rho(a) \frac{1-\epsilon}{2} F(z)$$

*be the characteristic homomorphisms of the corresponding family of bounded Fredholm modules  $\mathcal{E}(z)$  (1.5) and denote by  $\psi = \rho_0 * \rho_1 : qA \rightarrow \mathcal{O}$  their free product. Let finally  $\pi : \mathcal{T}A \rightarrow A$  be the canonical epimorphism. Then for  $R \gg 0$  sufficiently large the cochain*

$$\beta = \int_{\Gamma} \psi^*(\tau_{\mathcal{P}}(z^2)) d\mu(z) \in CC^0(q(A)) \quad (3.10)$$

*extends to an entire cyclic cocycle*

$$\beta \in CC_{\epsilon}^0(q_R(A)) \quad (3.11)$$

*which satisfies*

$$q_R(\pi)_* \circ \beta = \nu_0 \in CC_{\epsilon}^0(q_R(\mathcal{T}A)) \quad (3.12)$$

**Proof:** This is a consequence of the fact that the linear maps  $\rho_{0,1}^t : A \rightarrow \mathcal{O}$  of 3.4 are actually homomorphisms of algebras for  $t = 0$ . Therefore the characteristic map



factors as  $\Psi_0 : q(TA) \xrightarrow{q(\pi)} qA \xrightarrow{\psi} \mathcal{O}_K$ . The estimates used already in the proof of 3.5 yield then the desired result.  $\square$

We arrive now at the final formula for an entire cyclic cocycle representing the cohomology class of Connes' character 1.5. The original unbounded Fredholm module enters only through the characters of its associated holomorphic family of bounded Fredholm modules. This makes it very easy to compare Connes' character with various Chern-Connes characters of bounded Fredholm modules.

**Theorem 3.8.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . Let  $\mathcal{P} = (P_n)_{n \in \mathbb{N}}$  be a family of finite rank projections satisfying the conditions of 2.24 and let  $\tau_{\mathcal{P}}$  be the corresponding analytic cyclic cocycle over  $\mathcal{K}(\mathcal{H})$  (see 1.37).*

- a) *For  $R \gg 0$  sufficiently large denote by  $\iota_A^\epsilon : CC_*^\epsilon(A) \rightarrow CC_*^\epsilon(q_R A)$  a continuous chain map which represents the corresponding bivariate cohomology class in entire, analytic or local cyclic cohomology. (One can choose a chain map which is independent of  $R$  and natural in  $A$ ). Let  $\mathcal{E}' = (\mathcal{H}, \rho, \epsilon, F)$  be a bounded even Fredholm module over  $A$  and let  $\psi : q_R A \rightarrow q_{C^*} A \rightarrow \mathcal{L}(\mathcal{H})$  (1.34) be its characteristic homomorphism. Then the analytic cyclic cocycle*

$$\check{ch}_{\mathcal{P}}(\mathcal{E}') = \iota_A^\epsilon \circ \psi_* \circ \tau_{\mathcal{P}} \in CC_0^{\text{anal}}(A, \mathbb{C}) \quad (3.13)$$

*represents the abstract Chern-Connes character  $\check{ch}$  (1.27) in  $K$ -homology.*

- b) *Let  $\mathcal{E}(z) = (\mathcal{H}, \rho, \epsilon, F(z))$ ,  $z \in \mathbb{C} \setminus \mathbb{R}_-$  (1.5) be the holomorphic family of bounded Fredholm modules associated to the given  $\theta$ -summable unbounded Fredholm module  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  over  $A$ . Then the cochain*

$$Ch_{\mathcal{P}}(\mathcal{E}) = \frac{1}{\sqrt{\pi}i} \int_{1-i\infty}^{1+i\infty} \check{ch}_{\mathcal{P}}(\mathcal{E}(z^2)) e^{z^2} dz \quad (3.14)$$

*extends to an entire cyclic cocycle over  $A$ .*

- c) *The entire cyclic cocycle  $Ch_{\mathcal{P}}(\mathcal{E})$  represents the cohomology class of Connes' character  $Ch_\epsilon(\mathcal{E})$  1.5 in the entire cyclic cohomology  $HC_\epsilon^0(A)$  of  $A$ :*

$$[Ch_{\mathcal{P}}(\mathcal{E})] = [Ch_\epsilon(\mathcal{E})] \in HC_\epsilon^0(A) \quad (3.15)$$

**Proof:**

- a) The characteristic homomorphism (1.34)  $\psi : qA \rightarrow \mathcal{K}(\mathcal{H})$  extends to a continuous homomorphism  $\psi : q_R A \rightarrow \mathcal{K}(\mathcal{H})$  for  $R > \|F\|$ . The pullback of the analytic cyclic cocycle  $\vartheta$  along  $\psi$  and  $\iota_A^\epsilon$  yields therefore an analytic cyclic cocycle on  $A$ . As  $\tau_{\mathcal{P}}$  represents the canonical generator of  $HC_{loc}^0(\mathcal{K}(\mathcal{H}))$  in local cyclic cohomology by (1.36) the formula (1.39) implies that the analytic cocycle  $\check{ch}_{\mathcal{P}}$  represents the Chern character in  $K$ -homology.

b) By (3.14) and the Cauchy integral formula (see [Co], pp.541-542)

$$\begin{aligned} Ch_{\mathcal{P}}(\mathcal{E}) &= \frac{1}{\sqrt{\pi}i} \int_{1-i\infty}^{1+i\infty} \check{ch}_{\mathcal{P}}(\mathcal{E}(z^2)) e^{z^2} dz = \\ &= \iota_A^\epsilon \circ \frac{1}{\sqrt{\pi}i} \int_{1-i\infty}^{1+i\infty} \psi_* \circ \tau_{\mathcal{P}}(z^2) d\mu(z) = \iota_A^\epsilon \circ \beta \end{aligned}$$

and the latter cocycle is entire by 3.7.

c) Let  $\pi : \mathcal{T}A \rightarrow A$  be the canonical epimorphism. Then for  $R \gg 0$  sufficiently large

$$\pi_* \circ Ch_{\mathcal{P}}(\mathcal{E}) = \pi_* \circ \iota_A^\epsilon \circ \beta = \iota_{\mathcal{T}A}^\epsilon \circ q_R(\pi)_* \circ \beta = \iota_{\mathcal{T}A}^\epsilon \circ \nu_0$$

by 3.7. According to 3.5 and 3.6 the latter cocycle is cohomologous to  $\iota_{\mathcal{T}A}^\epsilon \circ \nu_1 = \iota_{\mathcal{T}A}^\epsilon \circ \tilde{\tau}_{\mathcal{P}}$  which finally is cohomologous to  $\pi_* \circ Ch_\epsilon(\mathcal{E})$  by 2.29. Therefore

$$\pi_* [Ch_{\mathcal{P}}(\mathcal{E})] = \pi_* [Ch_\epsilon(\mathcal{E})] \in HC_0^\epsilon(\mathcal{T}A, \mathbb{C})$$

The topological version of Goodwillie's theorem [Pu2], 5.1 asserts that the element  $\pi_* \in HC_0^\epsilon(\mathcal{T}A, A)$  is a bivariant entire cyclic cohomology equivalence. So we deduce from the previous equation

$$[Ch_{\mathcal{P}}(\mathcal{E})] = [Ch_\epsilon(\mathcal{E})] \in HC_\epsilon^0(A, \mathbb{C})$$

which was our claim.  $\square$

## 4 Comparison of characters

**Theorem 4.1.** *Let  $\mathcal{E} = (\mathcal{H}, \rho, \epsilon, \mathcal{D})$  be an even unbounded  $\theta$ -summable Fredholm module over  $A$ . Let  $\overline{A}$  be the enveloping  $C^*$ -algebra of  $A$  and denote by  $[\mathcal{E}] \in K^0(\overline{A})$  be the  $K$ -homology class of  $\mathcal{E}$ . Let  $[Ch_\epsilon(\mathcal{E})] \in HC_\epsilon^0(A)$  be the cohomology class of Connes' character cocycle 1.5 of  $\mathcal{E}$  and let  $\check{ch}([\mathcal{E}]) \in HC_{loc}^0(\overline{A})$  be the abstract Chern-Connes character (1.27) of the  $K$ -homology class of  $\mathcal{E}$ . Let finally  $HC_\epsilon \rightarrow HC_{loc}$  be the canonical natural transformation ([Pu2], section 7) from entire to local cyclic (co)homology. Then the images of the characters*

$$\begin{aligned} HC_\epsilon^0(A) &\rightarrow HC_{loc}^0(A) \leftarrow HC_{loc}^0(\overline{A}) \\ Ch_\epsilon(\mathcal{E}) &\rightarrow * \leftarrow \check{ch}([\mathcal{E}]) \end{aligned} \tag{4.1}$$

in  $HC_{loc}^0(A)$  coincide.

**Proof:** According to the previous theorem (see the proof of part b))

$$Ch_{\mathcal{P}}(\mathcal{E}) = \frac{1}{\sqrt{\pi}i} \int_{\Gamma} \check{ch}_{\mathcal{P}}(F(z^2)) d\mu(z)$$

Whereas only the whole integral defines an entire cyclic cocycle the individual terms  $\check{ch}_{\mathcal{P}}(F(z^2)) = \iota_A^\epsilon \circ \psi_* \circ \tau_{\mathcal{P}}(z^2)$  make sense as analytic cyclic cocycles. Variation of the parameter  $z \in \Gamma$  to  $z_0 = 1$  along  $\Gamma$  gives rise to a smooth homotopy of the characteristic homomorphisms (1.32). The Cartan homotopy formula in analytic cyclic cohomology [Pu], 5.21 shows then that

$$\check{ch}_{\mathcal{P}}(F(z^2)) = \check{ch}_{\mathcal{P}}(F(1)) + \partial(\chi(z^2)) \quad (4.2)$$

where  $\chi(z^2) \in CC_{anal}^1(A)$  is an analytic cyclic cochain over  $A$ . The growth conditions of the cochain  $\chi(z^2)$  are satisfied uniformly with respect to  $z \in \Gamma$  as a look at 2.26, 2.27 and the definition of  $\chi$  shows. Therefore

$$\chi = \frac{1}{\sqrt{\pi}i} \int_{\Gamma} \chi(z^2) d\mu(z)$$

extends to an analytic cyclic cochain on  $A$ . Integration of (4.2) over  $\Gamma$  provides the equality

$$Ch_{\mathcal{P}}(\mathcal{E}) = \check{ch}_{\mathcal{P}}(F(1)) \frac{1}{\sqrt{\pi}i} \int_{\Gamma} d\mu(z) + \partial(\chi) = \check{ch}_{\mathcal{P}}(F(1)) + \partial(\chi)$$

of analytic cyclic cocycles. If we pass thus from entire to analytic cyclic cohomology [Me],[Pu] we find that

$$[Ch_{\epsilon}(\mathcal{E})] = [Ch_{\mathcal{P}}(\mathcal{E})] = [\check{ch}_{\mathcal{P}}(F(1))] \in HC_{anal}^0(A)$$

coincide. Let  $\overline{A}$  be the enveloping  $C^*$ -algebra of  $A$  and consider the commutative diagram (in the notations of (1.33) and (2.43))

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{K}(\mathcal{H}) & \longrightarrow & \mathcal{L}(\mathcal{H}) & \longrightarrow & \mathcal{Q}(\mathcal{H}) \rightarrow 0 \\ & & \psi \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & q_{C^*} \overline{A} & \longrightarrow & Q_{C^*} \overline{A} & \longrightarrow & \overline{A} \rightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow i \\ 0 & \rightarrow & q_R A & \longrightarrow & Q_R A & \longrightarrow & A \rightarrow 0 \end{array}$$

which exists for all  $R \geq 1$  as homomorphisms of  $C^*$ -algebras are contractive. A look at the formula (3.14) for the Chern-Connes character shows that  $\check{ch}_{\mathcal{P}}(F(1)) \in CC_{anal}^0(A)$  extends to an analytic cyclic cocycle on the  $C^*$ -algebra  $\overline{A}$ . Viewed as a local cyclic cocycle it represents the Chern-Connes character  $\check{ch}(\mathcal{E}(1)) \in HC_{loc}^0(\overline{A})$  of the  $K$ -homology class of the bounded Fredholm module  $\mathcal{E}(1) = (\mathcal{H}, \rho, \epsilon, F(1))$ . Now by definition

$$F(1) = \frac{\mathcal{D} + \epsilon}{(1 + \mathcal{D}^2)^{\frac{1}{2}}} = f(\mathcal{D}) + \epsilon(1 + \mathcal{D}^2)^{-\frac{1}{2}}, \quad f(x) = x(1 + x^2)^{-\frac{1}{2}}$$

Therefore  $f(\mathcal{D})$  is a compact perturbation of  $F(1)$  which shows that this bounded Fredholm module represents the  $K$ -homology class of  $\mathcal{E}$ . So we deduce finally

$$\check{ch}([\mathcal{E}]) = [\check{ch}_{\mathcal{P}}(F(1))] = [Ch_{\mathcal{P}}(\mathcal{E})] = [Ch_{\epsilon}(\mathcal{E})] \in HC_{loc}^0(\overline{A})$$

which proves the theorem.  $\square$

## 5 On a problem of Connes

As an application of the comparison result for characters of Fredholm modules we present a partial solution of a problem posed by A. Connes in [Co3], p.83 (see also [Co2], pp.407-414). There he constructs a certain infinite dimensional unbounded  $\theta$ -summable Fredholm module over the reduced group  $C^*$ -algebra of a lattice in a semisimple Lie-group. The  $K$ -homology class of this module is closely related to Kasparov's  $\gamma$ -element [Ka1], [Co2]. Connes predicts that the character cocycle of this Fredholm module is cohomologous to the canonical trace and asks to verify this. We will determine the class of Connes' character cocycle in local cyclic cohomology for uniform lattices in semisimple Lie groups of real rank one. This is as far as I know the first calculation of the Connes character of an infinite dimensional unbounded  $\theta$ -summable Fredholm module.

We begin by a presentation of Connes' problem.

**Definition 5.1.** [Co3],pp.81-82, [Co2],pp.408-409.

Let  $\Gamma$  be a discrete subgroup of a real semisimple Lie group  $G$ . Let  $G/K$  be the associated symmetric space and let  $d(-, -)$  be the distance function associated to a (left)-translation invariant Riemannian metric on  $G/K$ . Let  $L^2(G/K, \Lambda^*)$  be the Hilbert space of square-integrable differential forms graded by the decomposition into forms of even and odd degree, respectively. Fix a base point  $x_0 \in G/K$ . Then

$$\mathcal{D}_{(G, x_0)} = e^\varphi \cdot d \cdot e^{-\varphi} + (e^\varphi \cdot d \cdot e^{-\varphi})^*, \quad \varphi(x) = d(x, x_0)^2 \quad (5.1)$$

is a densely defined, unbounded, odd, essentially selfadjoint operator on  $L^2(G/K, \Lambda^*)$  and

$$\mathcal{E}_\Gamma = (L^2(G/K, \Lambda^*), \mathcal{D}_{(G, x_0)}) \quad (5.2)$$

defines a  $\theta$ -summable unbounded even Fredholm module over a dense and holomorphically closed Banach subalgebra  $\mathcal{A}(\Gamma) \subset C_r^*(\Gamma)$  of the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  of  $\Gamma$ .

This Fredholm module cannot be finitely summable according to [Co5] because  $G$  contains non-amenable discrete subgroups.

The problem which Connes poses in [Co3], p.83 is equivalent to the

**Problem 5.2. (Connes)**

*Show that the Connes character of  $\mathcal{E}_\Gamma$  is cohomologous to the restriction of the canonical trace  $\tau_{can}$  on  $C_r^*(\Gamma)$  to  $\mathcal{A}(\Gamma)$ :*

$$[Ch_\epsilon(\mathcal{E}_\Gamma)] = [\tau_{can}] \in HC_\epsilon^0(\mathcal{A}(\Gamma)) \quad (5.3)$$

The interest in this problem stems from the fact that a positive answer provides a solution of the Kadison-Kaplansky idempotent conjecture for the considered group.

We recall what is known about the  $K$ -homology class of  $\mathcal{E}_\Gamma$ . The next lemma collects material from [Ka1] and [Co2].

**Lemma 5.3.** *Let  $\mathcal{E}_\Gamma$  be the unbounded Fredholm module introduced in 5.1 and let  $[\mathcal{E}_\Gamma] \in K^0(C_r^*(\Gamma))$  be its associated  $K$ -homology class. Denote by  $p : C_{max}^*(\Gamma) \rightarrow C_r^*(\Gamma)$  the canonical epimorphism. Let*

$$KK(- \rtimes_{max} \Gamma, \mathbb{C}) \simeq KK^\Gamma(-, \mathbb{C}) \quad (5.4)$$

*be the tautological isomorphism. In particular*

$$KK(C_{max}^* \Gamma, \mathbb{C}) \simeq KK^\Gamma(\mathbb{C}, \mathbb{C})$$

*Then the pullback  $p_* \circ [\mathcal{E}_\Gamma] \in KK_0(C_{max}^*(\Gamma), \mathbb{C})$  corresponds under the tautological isomorphism to Kasparov's canonical  $\gamma$ -element  $\gamma_\Gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$*

$$\begin{array}{ccccc} KK(C_r^*(\Gamma), \mathbb{C}) & \longrightarrow & KK(C_{max}^*(\Gamma), \mathbb{C}) & \simeq & KK^\Gamma(\mathbb{C}, \mathbb{C}) \\ [\mathcal{E}_\Gamma] & \longrightarrow & p_* \circ [\mathcal{E}_\Gamma] & \leftrightarrow & \gamma_\Gamma \end{array} \quad (5.5)$$

The abstract characterization of the considered Fredholm module will allow us to apply, according to the comparison result of the previous section, our partial calculation of the abstract Chern-Connes character of the  $\gamma$ -element in [Pu3]. This will lead to the promised partial solution of Connes' problem.

In the next step we clarify the behaviour of the tautological isomorphism (5.4) with respect to the various bivariant Chern-Connes characters.

**Lemma 5.4.** *Let  $\Gamma$  be a countable discrete group, let  $A$  be a separable  $\Gamma$ - $C^*$ -algebra and let  $A \rtimes_{max} \Gamma$  be the associated maximal crossed product  $C^*$ -algebra. Let  $\iota : \ell^1(\Gamma, A) \rightarrow A \rtimes_{max} \Gamma$  be the canonical inclusion and denote by  $\pi : \ell^1(\Gamma) \rightarrow \mathbb{C}$  the trivial representation. Let*

$$ch_{biv} : KK_*(-, -) \longrightarrow HC_*^{loc}(-, -) \quad (5.6)$$

*be the bivariant Chern-Connes character [Pu1], section 6 and let*

$$ch_{\ell^1(\Gamma)}^\Gamma : KK^\Gamma(-, -) \longrightarrow HC_0^{loc}(\ell^1(\Gamma, -), \ell^1(\Gamma, -)) \quad (5.7)$$

*be the equivariant bivariant Chern-Connes character [Pu3], 5.2. Then there is a natural commutative diagram*

$$\begin{array}{ccc} KK^\Gamma(A, \mathbb{C}) & \simeq & KK(A \rtimes_{max} \Gamma, \mathbb{C}) \\ & & \downarrow ch_{biv} \\ ch_{\ell^1(\Gamma)}^\Gamma \downarrow & & HC_0^{loc}(A \rtimes_{max} \Gamma, \mathbb{C}) \\ & & \downarrow \iota_* \circ - \end{array} \quad (5.8)$$

$$HC_0^{loc}(\ell^1(\Gamma, A), \ell^1(\Gamma)) \xrightarrow{- \circ \pi_*} HC_0^{loc}(\ell^1(\Gamma, A), \mathbb{C})$$

**Proof:**

Observe that the diagram is natural in the following sense. For every  $\Gamma$ - $C^*$ -algebra  $B$  the abelian group  $KK^\Gamma(B, A)$  acts on the groups in the diagram by left multiplication, and all arrows in the diagram are compatible with this action. In fact, any element  $\alpha \in KK^\Gamma(B, A)$  acts via left multiplication by  $j(\alpha) \in KK(B \rtimes_{\max} \Gamma, A \rtimes_{\max} \Gamma)$  on  $KK(A \rtimes_{\max} \Gamma, -)$ , via  $ch_{biv}(j(\alpha))$  on  $HC_0^{loc}(A \rtimes_{\max} \Gamma, -)$ , and via  $ch_{\ell^1(\Gamma)}^\Gamma(\alpha)$  on  $HC_0^{loc}(\ell^1(\Gamma, A), -)$ . The compatibility of these actions with the arrows in the diagram is equivalent to the multiplicativity of the descent homomorphism  $j$  [Ka1] and the multiplicativity of the various bivariate Chern-Connes characters [Pu1], [Pu3], respectively. Every element  $\alpha \in KK^\Gamma(A, \mathbb{C})$  can be written as a product  $\alpha = \alpha \otimes 1_\Gamma$  where  $1_\Gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$  is the unit. Therefore it suffices to verify that the diagram commutes when applied to the unit class  $1_\Gamma \in KK^\Gamma(\mathbb{C}, \mathbb{C})$ . Now the unit class corresponds under the tautological isomorphism (5.4) to the trivial representation of  $C_{\max}^*(\Gamma)$ . The assertion follows then from the naturality of the Chern-Connes characters. □

The main result of [Pu3] yields by the previous lemma

**Theorem 5.5.** *Let  $\Gamma$  be a torsion free, cocompact, discrete subgroup of a semisimple Lie group  $G$  of real rank one. Let*

$$[\mathcal{E}_\Gamma] \in K^0(C_r^*(\Gamma)) = KK(C_r^*(\Gamma), \mathbb{C})$$

*be the  $K$ -homology class of Connes' Fredholm module  $\mathcal{E}_\Gamma$  introduced in 5.1. Then its abstract Chern-Connes character is given by the cohomology class of the canonical trace*

$$\check{ch}([\mathcal{E}_\Gamma]) = [\tau_{can}] \in HC_{loc}^0(C_r^*(\Gamma)) \quad (5.9)$$

**Proof:** We proceed in several steps.

1) According to our assumptions the space  $\Gamma \backslash G / K$  is a compact Riemannian manifold of negative sectional curvature. Therefore the group  $\Gamma$  is word-hyperbolic. In particular, it satisfies Jolissaint's condition of rapid decay [CM], section 6. This means that the space  $\mathfrak{A}(\Gamma)$  of square-summable functions of rapid decay (w.r.t. a word metric) on  $\Gamma$  forms an algebra under convolution which is dense and holomorphically closed in  $C_r^*(\Gamma)$ .

2) It was shown in [Pu3] that for a word-hyperbolic group  $\Gamma$  the local cyclic cohomology groups of the algebras  $\ell^1(\Gamma)$ ,  $\mathfrak{A}(\Gamma)$  and of  $\ell^1(\Gamma) \cap \mathfrak{A}(\Gamma)$  decompose as the direct sum of a homogeneous and an inhomogeneous part. The homogeneous part corresponds to the contribution of the conjugacy class of the unit and the inhomogeneous part to the contribution of the other conjugacy classes of  $\Gamma$ . Moreover the various inclusions of these algebras induce isomorphisms

$$HC_{loc}^*(\mathfrak{A}(\Gamma))_{hom} \xrightarrow{\cong} HC_{loc}^*(\ell^1(\Gamma) \cap \mathfrak{A}(\Gamma))_{hom} \xleftarrow{\cong} HC_{loc}^*(\ell^1(\Gamma))_{hom} \quad (5.10)$$

of the homogeneous parts of the corresponding local cyclic cohomology groups.

3) Recall that by definition the abstract character  $\check{ch}$  in  $K$ -homology corresponds to the bivariant Chern-Connes character  $ch_{biv}$  under the identification  $K^*(-) = KK_*(-, \mathbb{C})$ . We calculate the pull back of  $\check{ch}([\mathcal{E}_\Gamma]) \in HC_{loc}^0(C_r^*(\Gamma))$  along the inclusion  $i : \ell^1(\Gamma) \rightarrow C_r^*(\Gamma)$ . Lemma 5.3 and the commutative diagram of lemma 5.4 show that

$$i_* \circ \check{ch}([\mathcal{E}_\Gamma]) = \iota_* \circ p_* \circ \check{ch}([\mathcal{E}_\Gamma]) = \iota_* \circ \check{ch}(p^*[\mathcal{E}_\Gamma]) = ch_{\ell^1(\Gamma)}^\Gamma(\gamma_\Gamma) \circ \pi \quad (5.11)$$

In [Pu3] we gave a qualitative characterization of the equivariant Chern-Connes character of the  $\gamma$ -element for a torsion free word-hyperbolic group:

$$ch_{\ell^1(\Gamma)}^\Gamma(\gamma_\Gamma) = \pi_{Hom} \in HC_0^{loc}(\ell^1(\Gamma), \ell^1(\Gamma)) \quad (5.12)$$

where  $\pi_{Hom}$  denotes the canonical projection onto the homogeneous part of the local cyclic complex of  $\ell^1(\Gamma)$ . So we conclude

$$i_* \circ \check{ch}([\mathcal{E}_\Gamma]) = \pi_{Hom} \circ \pi_* = i_* \circ [\tau_{can}] \in HC_{loc}^0(\ell^1(\Gamma)) \quad (5.13)$$

4) Let  $i' : \mathfrak{A}(\Gamma) \rightarrow C_r^*(\Gamma)$  be the canonical inclusion. By 2) we have

$$i'_* \circ \check{ch}([\mathcal{E}_\Gamma]) = (i'_* \circ \check{ch}([\mathcal{E}_\Gamma]))_{hom} + (i'_* \circ \check{ch}([\mathcal{E}_\Gamma]))_{inhom} \in HC_{loc}^0(\mathfrak{A}(\Gamma))$$

and from 2) and 3) we learn

$$(i'_* \circ \check{ch}([\mathcal{E}_\Gamma]))_{hom} = i'_* \circ [\tau_{can}] \in HC_{loc}^0(\mathfrak{A}(\Gamma))_{hom} \quad (5.14)$$

5) We calculate the inhomogeneous part of  $i'_* \circ \check{ch}([\mathcal{E}_\Gamma])$ . Recall that the  $\gamma$ -element appears as a Kasparov product

$$\gamma = \beta \otimes \alpha, \beta \in KK^\Gamma(\mathbb{C}, C_0(X)), \alpha \in KK^\Gamma(C_0(X), \mathbb{C}) \quad (5.15)$$

where  $X$  is a manifold on which  $\Gamma$  acts properly and freely. Because the action of  $\Gamma$  on  $X$  is proper, the associated maximal and reduced crossed products coincide:

$$C_0(X) \rtimes_{max} \Gamma \xrightarrow{\simeq} C_0(X) \rtimes_r \Gamma \quad (5.16)$$

The compatibility of the tautological isomorphism (5.4) with the Kasparov product, discussed in the proof of 5.4, shows that

$$[\mathcal{E}_\Gamma] = j_r(\beta) \otimes \alpha', j_r(\beta) \in KK(C_r^*\Gamma, C_0(X) \rtimes_r \Gamma), \alpha' \in KK(C_0(X) \rtimes_r \Gamma, \mathbb{C})$$

where the class  $\alpha' \in KK(C_0(X) \rtimes_r \Gamma, \mathbb{C}) = KK(C_0(X) \rtimes_{max} \Gamma, \mathbb{C})$  corresponds to  $\alpha \in KK^\Gamma(C_0(X), \mathbb{C})$  under (5.4). The multiplicativity of the Chern-Connes character implies

$$i'_* \circ \check{ch}([\mathcal{E}_\Gamma]) = i'_* \circ ch_{biv}(j_r(\beta)) \circ \check{ch}(\alpha') = ch_{\mathfrak{A}(\Gamma)}^\Gamma(\beta) \circ i'_* \circ \check{ch}(\alpha') \quad (5.17)$$

$ch_{\mathfrak{A}(\Gamma)}^\Gamma(\beta) \in HC_0^{loc}(\mathfrak{A}(\Gamma), \mathfrak{A}(\Gamma, C_0(X)))$ ,  $i'_* \circ \check{ch}(\alpha') \in HC_{loc}^0(\mathfrak{A}(\Gamma, C_0(X)))$ , and similar identities hold for the homogeneous and inhomogeneous parts [Pu3], 5.2, respectively. It has been shown in [Pu3], 4.8 that

$$HC_{loc}^0(\mathfrak{A}(\Gamma, C_0(X)))_{inhom} = 0 \quad (5.18)$$

because  $\Gamma$  acts properly on  $X$ . We deduce from this and the previous identity that

$$i'_* \circ \check{ch}(\alpha')_{inhom} = 0 \quad \text{and} \quad i'_* \circ \check{ch}([\mathcal{E}_\Gamma])_{inhom} = 0 \quad (5.19)$$

6) From parts 4) and 5) we may conclude that

$$i'_* \circ \check{ch}([\mathcal{E}_\Gamma]) = i'_* \circ \tau_{can} \in HC_{loc}^0(\mathfrak{A}(\Gamma)) \quad (5.20)$$

Cowling and Haagerup have shown [CH], that the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  of a cocompact discrete subgroup of a semisimple Lie group of real rank one possesses the Grothendieck approximation property. This implies that the inclusion of any dense subalgebra  $\mathcal{R} \hookrightarrow C_r^*(\Gamma)$ , which is the domain of an unbounded derivation on  $C_r^*(\Gamma)$ , induces a local cyclic cohomology equivalence [Pu3], 5.15. and 3.10. The Jolissaint algebra is of this type so that we deduce that

$$i'_* \in HC_0^{loc}(\mathfrak{A}(\Gamma), C_r^*(\Gamma)) \quad (5.21)$$

is a local cyclic cohomology equivalence for the groups under consideration. Thus the identity (5.20) implies

$$\check{ch}([\mathcal{E}_\Gamma]) = [\tau_{can}] \in HC_{loc}^0(C_r^*(\Gamma)) \quad (5.22)$$

which was to be proved.  $\square$

As a consequence we obtain finally a partial solution of Connes' problem.

**Theorem 5.6.** *Let  $\Gamma$  be a torsion free, cocompact, discrete subgroup of a semisimple Lie group  $G$  of real rank one. Let*

$$\mathcal{E}_\Gamma = (L^2(G/K, \Lambda^*), \mathcal{D}_{(G, x_0)})$$

*be Connes' unbounded  $\theta$ -summable Fredholm module over the dense and holomorphically closed Banach subalgebra  $\mathcal{A}(\Gamma) \subset C_r^*(\Gamma)$ . Then Connes' character cocycle*

$$Ch_\epsilon(\mathcal{E}_\Gamma) \in CC_\epsilon^0(\mathcal{A}(\Gamma))$$

*is cohomologous in local cyclic cohomology to the canonical trace on  $\mathcal{A}(\Gamma)$ :*

$$[Ch_\epsilon(\mathcal{E}_\Gamma)] = [\tau_{can}] \in HC_{loc}^0(\mathcal{A}(\Gamma)) \quad (5.23)$$



**Proof:** Let  $j : \mathcal{A}(\Gamma) \rightarrow C_r^*(\Gamma)$  be the inclusion. According to the comparison theorem 4.1 and theorem 5.5 the image of Connes' character cocycle  $Ch_\epsilon(\mathcal{E}_\Gamma)$  in local cyclic cohomology satisfies

$$[Ch_\epsilon(\mathcal{E}_\Gamma)] = j_* \circ \check{ch}([\mathcal{E}_\Gamma]) = j_* \circ [\tau_{can}] = [\tau_{can}] \in HC_{loc}^0(\mathcal{A}(\Gamma)) \quad (5.24)$$

□

As a well known consequence (see [Co1], [Co2]) we note the following result which was obtained previously with other methods by Lafforgue [La], Mineyev-Yu [MY], Kasparov-Yu (unpublished), and myself [Pu3].

**Theorem 5.7.** (*Kadison-Kaplansky conjecture*)

*Let  $\Gamma$  be a torsion free, cocompact, discrete subgroup of a semisimple Lie group  $G$  of real rank one. Then the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$  contains no idempotents except 0 and 1.*

**Proof:** (following Connes) By a well known argument it suffices to show that the canonical trace takes only integer values on idempotents in  $C_r^*(\Gamma)$ . Let  $e = e^2 \in \mathcal{A}(\Gamma)$  and let  $[e] \in K_0(\mathcal{A}(\Gamma))$  be its class in  $K$ -theory. The index formula 1.6 and the previous theorem show that

$$\tau_{can}(e) = \langle [\tau_{can}], ch([e]) \rangle = \langle [Ch_\epsilon(\mathcal{E}_\Gamma)], ch([e]) \rangle = Ind(\mathcal{E}_\Gamma \otimes [e]) \in \mathbb{Z} \quad (5.25)$$

Because  $\mathcal{A}(\Gamma)$  is dense and holomorphically closed in  $C_r^*(\Gamma)$  the same conclusion holds for idempotents  $e = e^2 \in C_r^*(\Gamma)$ . This implies the theorem. □

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